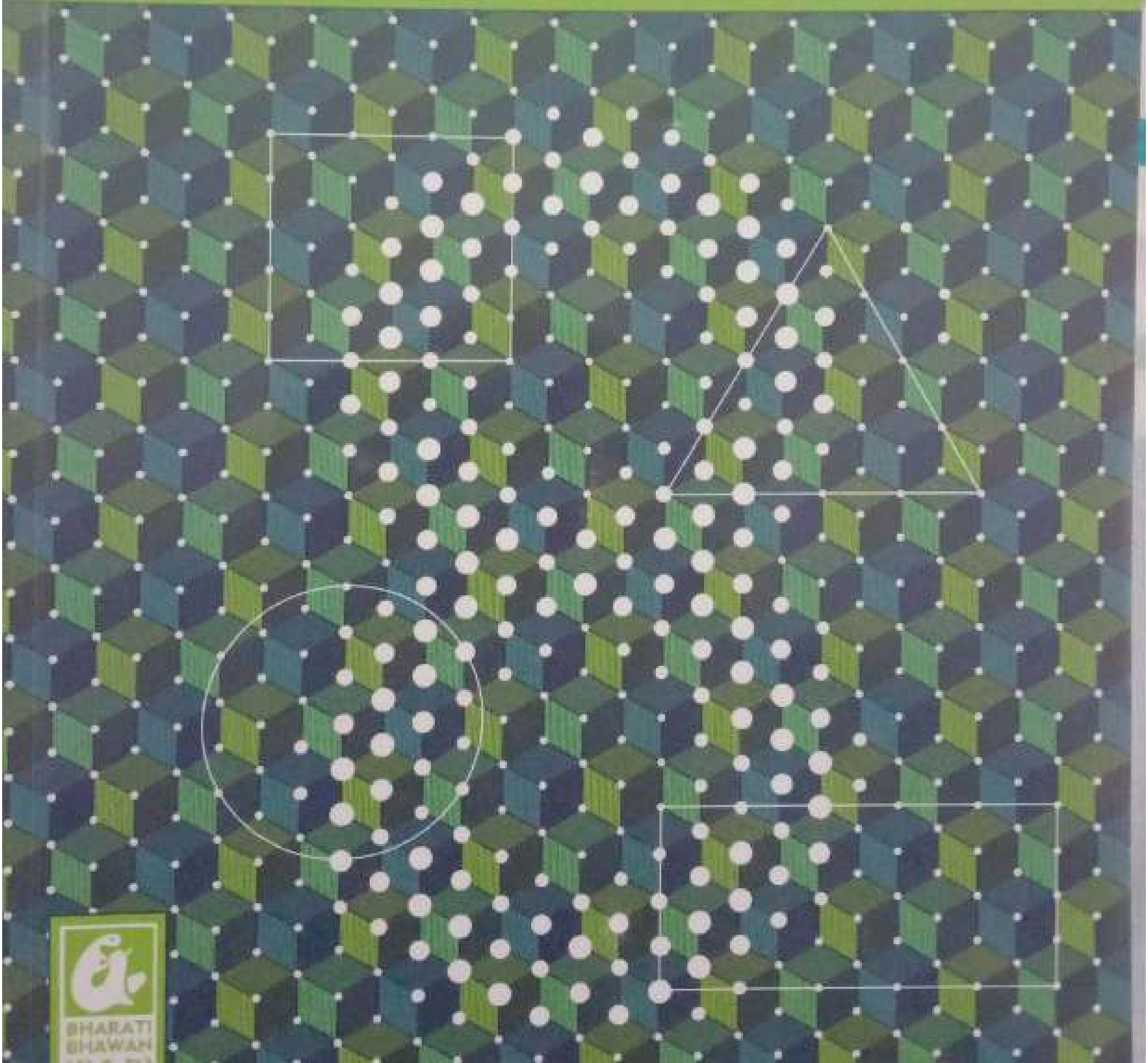
ESSENTIAL ICSE

MATHEMATICS

FOR CLASS 8



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1
Idea of Sets

Sets

You are already familiar with some basic concepts of sets. Let us review what you have learnt and then move on to some new concepts.

A set is a collection of well-defined, distinct objects. The objects of a set are called the members, or elements, of the set. We call a set a 'well-defined' collection of objects because we can decide with absolute certainty whether a given object is a member of the set.

Examples (i) The set of all integers

(ii) The set of the vowels of the English alphabet

(iii) The set of the rivers of India

(iv) The set of the students of Class VIII of your school

We usually denote a set by a capital letter of the English alphabet, such as A, B, C, X, Y and Z, and its elements by small letters of the English alphabet, such as a, b, c, x and y.

If a is an element of the set X, we write $a \in X$ and read this as "a belongs to the set X." If x is not an element of the set A, we write $x \notin A$ and read this as "x does not belong to the set A."

Representation of sets

A set is usually represented either in the tabular form or in the set-builder form.

Roster method (or tabular form)

In this method, we represent a set by listing all its elements between braces, or second (curly) brackets.

Examples (i) The set A of all the vowels of the English alphabet is represented as $A = \{a, e, i, o, u\}$.

(ii) The set X of all the days of a week is represented as X = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}.

The order in which we list the elements of a set within braces is immaterial. Thus, each of the following denotes the same set:

$$\{a, b, c\}, \{b, a, c\}, \{a, c, b\}, \{c, a, b\}, \{c, b, a\}, \{b, c, a\}.$$

In listing the elements of a set in the tabular form, we do not repeat any element. Thus, if B is the set of all the digits in the number 15 312 142 then $B = \{1, 5, 3, 2, 4\}$.

Rule method (or set-builder form)

We can represent a set by stating a property which its elements satisfy. Thus, the set of all natural numbers less than 10 is

$A = \{x \mid x$ is a natural number, where $x < 10\}$.

or $A = \{x : x$ is a natural number and $x < 10\}$.

We read this as: "A is the set of all elements x such that x is a natural number and x is less than 10."

Examples (i) The set $X = \{2, 4, 6, 8\}$ can be written in the set-builder form as

$X = \{x \mid x$ is an even natural number and $x \le 8\}$.

(ii) The set of the prime numbers that are less than 15 can be written in the set-builder form as

$A = \{x \mid x$ is a prime number and $x < 15\}$.

The set A is written in the tabular form as $A = \{2, 3, 5, 7, 11, 13\}$.

We sometimes represent a set by describing a property of its elements inside braces.

Example Suppose A = {days of a week} and B = {one-digit odd numbers}.

In the tabular form,

A = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

and $B = \{1, 3, 5, 7, 9\}$.

Some special sets

(i) $N = \{1, 2, 3, 4, \ldots\}$ is the set of natural numbers.

(ii) $W = \{0, 1, 2, 3, \ldots\}$ is the set of whole numbers.

(iii) $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of integers. This is also denoted by I.

Solved Examples

EXAMPLE 1 **Write each of the following sets in the roster form.**

(i) The set of the months in a year that end in the letter 'y'

(ii) The set of the months in a year that have 31 days

(iii) The set of the odd numbers between 10 and 20

Solution (i) {January, February, May, July}

(ii) {January, March, May, July, August, October, December}

(iii) {11, 13, 15, 17, 19}

EXAMPLE 2 **Express each of the following sets in the set-builder form.**

(i) The set of the prime numbers between 20 and 30

(ii) The set of the whole numbers which are divisible by 5 and are less than 35

(iii) The set of the factors of 25

Solution (i) $\{x : x \text{ is a prime number, where } 20 < x < 30\}$.

(ii) $\{x : x = 5n, \text{ where } n \in W \text{ and } n < 7\}$.

(iii) $\{x \mid x \text{ is a factor of } 25\}$.

EXAMPLE 3 **Write each of the following sets in the roster form.**

(i) $\left\{x : x = \dfrac{n}{2n+1}, \text{ where } n \in N \text{ and } n < 4\right\}$

(ii) $\{x \mid 5x + 3 < 24, \text{ where } x \in W\}$

(iii) $\{x : x = 2r + 3, \text{ where } r \in I \text{ and } -2 < r \le 3\}$

Solution (i) Given, $n \in N$ and $n < 4$. So, $n = 1, 2, 3$. Also, $x = \dfrac{n}{2n+1}$.

Substituting $n = 1, 2, 3$, we get $x = \dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}$ respectively.

$\therefore$ the given set is $\left\{\dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}\right\}$.

(ii) Here, $5x + 3 < 24$. So, $5x < 21$ or $x < \dfrac{21}{5}$.

Since $x \in W$, we get $x = 0, 1, 2, 3, 4$.

So, the given set is $\{0, 1, 2, 3, 4\}$.

(iii) Given, $-2 < r \le 3$ and $r \in I$.

So, $r = -1, 0, 1, 2, 3$. Also, $x = 2r + 3$.

Substituting $r = -1, 0, 1, 2, 3$, we get $x = 1, 3, 5, 7, 9$ respectively.

Hence, the given set is $\{1, 3, 5, 7, 9\}$.

EXAMPLE 4 **Express each of the following sets in the tabular and set-builder forms.**

(i) **The set E of even natural numbers** (ii) **The set X of the factors of 36**

Solution (i) $E = \{2, 4, 6, 8, \ldots\}$ (tabular form)

$= \{x : x = 2n, n \in N\}$ (set-builder form).

(ii) $X = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ (tabular form)

$= \{x : x \text{ is a factor of } 36\}$ (set-builder form).

EXAMPLE 5 **Express each of the following sets in the set-builder form.**

(i) $\left\{\dfrac{4}{5}, \dfrac{5}{6}, \dfrac{6}{7}, \dfrac{7}{8}, \dfrac{8}{9}, \dfrac{9}{10}\right\}$ (ii) $\{0, 2, 4, 6, 8, 10, 12\}$

(iii) $\left\{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}, \dfrac{1}{64}\right\}$ (iv) $\{-40, -35, -30, \ldots, 20\}$

Solution (i) $\left\{x : x = \dfrac{n}{n+1}, \text{ where } n \in N \text{ and } 4 \le n \le 9\right\}$

(ii) $\{x \mid x = 2n, \text{ where } n \in W \text{ and } n \le 6\}$

(iii) $\left\{x : x = \dfrac{1}{2^m}, \text{ where } m \in N \text{ and } m \le 6\right\}$

(iv) $\{x \mid x = 5p, \text{ where } p \in I \text{ and } -8 \le p \le 4\}$

Remember These

1. A collection of distinct objects is a set if one can decide with absolute certainty whether a particular object is a member of the collection.
2. In the roster method (tabular form), the elements of a set are listed between braces, i.e., inside { }.
3. In the rule method (set-builder form), the elements are described or defined by a common property of the members.
4. No element of a set is repeated while writing a set by the roster method.

EXERCISE 1A

1. Which of the following collections are sets?
   (i) The collection of the positive integers that are less than 6
   (ii) The collection of big cities of India
   (iii) The collection of rich people in India
   (iv) The collection of the integers that are divisible by 3
2. If $A = \{1, 2, 3, 4, 5, 6\}$ then which of the following statements are true?
   (i) $6 \in A$ (ii) $7 \in A$ (iii) $2 \in A, 3 \in A$ and $5 \in A$
   (iv) $\{1, 2, 3\} \in A$ (v) $6 \in A$ and $8 \in A$
3. Write each of the following sets by the roster method.
   (i) The set of the last three months of a year (ii) The set of the angles of $\triangle ABC$
   (iii) The set of the even integers between 25 and 35
4. Express each of the following sets in the set-builder form.
   (i) The set of the even integers between 11 and 21
   (ii) The set of the whole numbers that are divisible by 6 and are less than 48
   (iii) The set of the factors of 30
5. Express each of the following sets in the tabular and set-builder forms.
   (i) The set of positive integers (ii) The set of odd natural numbers
   (iii) The set of the factors of 24 (iv) The set of the prime factors of 48
   (v) The set of the natural numbers which are perfect squares and are less than 50
6. Write the following sets in the tabular form.
   (i) $\left\{x \mid x = \frac{n}{n+1}, \text{ where } n \in N \text{ and } n \le 9\right\}$
   (ii) $\{p : p = 3n + 1, \text{ where } n \in W \text{ and } 2 \le n \le 5\}$
   (iii) $\{r : 7r + 5 < 36, \text{ where } r \in W\}$
   (iv) $\{x : x = y + 3, \text{ where } 10 \le y \le 14 \text{ and } y \in N\}$
   (v) $\{p : p = 7m, \text{ where } m \in I \text{ and } -1 \le m \le 3\}$
   (vi) $\{x \mid x \in N \text{ and } x^3 < 30\}$

7. Express the following sets in the set-builder form.

(i) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ (ii) {3, 6, 9, 12, 15} (iii) {0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40}

(iv) {1, 2, 13, 26} (v) {−24, −12, 0, 12, 24, 36, 48, 60}

ANSWERS

1. (i) and (iv)
2. (i), (ii) and (iii)
3. (i) {October, November, December} (ii) {∠BAC, ∠ABC, ∠ACB} (iii) {26, 28, 30, 32, 34}
4. (i) $\{x \mid x = 2n$, where $n \in I$ and $6 \le n \le 10\}$ (ii) $\{x : x = 6p$, where $p \in W$ and $p < 8\}$
(iii) $\{x : x$ is a factor of $30\}$
5. (i) {1, 2, 3, ...}; $\{x \mid x \in I$ and $x > 0\}$ (ii) {1, 3, 5, ...}; $\{x : x = 2n - 1$ and $n \in N\}$
(iii) {1, 2, 3, 4, 6, 8, 12, 24}; $\{x \mid x$ is a factor of $24\}$ (iv) {2, 3}; $\{x \mid x$ is a prime factor of $48\}$
(v) {1, 4, 9, 16, 25, 36, 49}; $\{x : x = n^2, x \in N$ and $n \le 7\}$
6. (i) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}\right\}$ (ii) {7, 10, 13, 16} (iii) {0, 1, 2, 3, 4} (iv) {13, 14, 15, 16, 17}
(v) {−7, 0, 7, 14, 21} (vi) {1, 2, 3}
7. (i) $\left\{x : x = \frac{1}{r}\text{, where } r \in N\right\}$ (ii) $\{x \mid x = 3n$, where $n \in N$ and $n \le 5\}$
(iii) $\{x : x = 4p$, where $p \in W$ and $p < 11\}$
(iv) $\{x : x \in N$ and x is a factor of $26\}$ (v) $\{x \mid x = 12n$, where $n \in I$ and $-2 \le n \le 5\}$

Types of Sets

Finite set

If the number of elements of a set is finite, the set is called a finite set.

Examples (i) The set of all the days of a week

(ii) $X = \{x : x$ is a factor of $6\}$

(iii) The set of the even numbers between 3 and 9

Infinite set

A set that does not have a fixed number of elements is called an infinite set. Such a set has an uncountable number of elements.

Examples (i) The set of all positive even integers

(ii) The set of integers, i.e., $I = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

(iii) The set of the natural numbers that are greater than 6, i.e., $\{x : x \in N$ and $x > 6\}$

Empty set

The set which contains no elements is called the empty set. This set is denoted by {} or Ø. The empty set is also called the null set or the void set.

Examples (i) The set of all women who are 6 m tall

(ii) The set of the odd integers with 6 as a factor

(iii) The set of the even prime numbers that are greater than 2

Note The number of elements in the empty set is 0. But {0} is not the empty set because it contains the element 0.

Singleton

A set that contains only one element is called a singleton.

Examples Each of the sets {0}, {12} and {a} is a singleton.

Cardinal number of a finite set

The cardinal number of a finite set A is the number of distinct elements in the set A. It is denoted by $n(A)$.

Examples (i) A = {months of a year}.

There are 12 months in a year.

Hence, $n(A) = 12$, i.e., the cardinal number of the set A is 12.

(ii) If $A = \{x, y, z\}$ and $B = \{p, q, r, s\}$ then $n(A) = 3$ and $n(B) = 4$.

Notes
- It is not possible to define the cardinal number of an infinite set.
- The cardinal number of the empty set (∅) is zero, as ∅ has no elements; i.e., $n(\emptyset) = 0$.
- The cardinal number of a singleton is 1. For example, if A = {present prime minister of India} then $n(A) = 1$.

Universal set

The set of all the possible objects (or elements) under consideration for a particular discussion is called the universal set. It is denoted by U or ξ. The universal sets for different problems may be different.

Example Let A = {students of your class who play badminton}

and B = {students of your class who play cricket}.

If our study is regarding the sets A and B, we may take the set of all the students of your class as the universal set. We may also take the set of all the students of your school as the universal set. However, in a particular problem there must be only one universal set.

Equivalent sets

Two finite sets are called equivalent sets if they contain the same number of elements. In other words, the sets A and B are equivalent sets if $n(A) = n(B)$. We express this in symbols as $A \sim B$ or $A \leftrightarrow B$.

Example The sets $A = \{2, 5, 8, 12\}$ and $B = \{4, 6, 9, 11\}$ are equivalent sets because $n(A) = n(B) = 4$. In symbols, $A \sim B$ or $A \leftrightarrow B$.

Subsets and Supersets

Subset

If A and B are two sets such that every element of A is an element of B, we say that "A is a subset of B." or "A is included in B." We express this in symbols as $A \subseteq B$. If the set A is not a subset of the set B, we write $A \nsubseteq B$.

Example Let $A = \{4, 5\}$ and $B = \{3, 4, 5\}$. Then, $4 \in A$ and $4 \in B$. Also, $5 \in A$ and $5 \in B$. Hence, every element of A is an element of B. So, $A \subseteq B$.

However, $3 \in B$ but $3 \notin A$. So, every element of B is not an element of A. Hence, B is not a subset of A; that is, $B \nsubseteq A$.

Notes • Each set is a subset of itself. Thus, for any set A, we can write $A \subseteq A$.

• The null set ($\emptyset$) is a subset of any set.

EXAMPLE **(i) Find all the subsets of the set $A = \{0, 1\}$.**

(ii) Write all the subsets of the set $P = \{1, 2, 3\}$.

(iii) Find all the subsets of the set $X = \{1, 2, 3, 4\}$.

Solution (i) The subsets of A containing just one element are $\{0\}$ and $\{1\}$. Also, the null set ($\emptyset$) and the set A itself are subsets of A.

$\therefore$ the subsets of the set A are $\emptyset$, $\{0\}$, $\{1\}$ and $\{0, 1\}$.

(ii) The subsets of P containing only one element are $\{1\}$, $\{2\}$ and $\{3\}$.

The subsets of P containing only two elements are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$.

The null set ($\emptyset$) and the set P itself are also subsets of the set P.

So, the subsets of the set P are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.

(iii) The subsets of X containing just one element are $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$.

The subsets of X containing only two elements are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$ and $\{3, 4\}$.

The subsets of X containing only three elements are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

The null set ($\emptyset$) and the set X itself are also subsets of X.

$\therefore$ the subsets of X are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$.

In the above examples we have considered the following.

(i) The number of subsets of the set A is $4 = 2^2 = 2^{\text{number of elements of } A}$.

(ii) The number of subsets of the set P is $8 = 2^3 = 2^{\text{number of elements of } P}$.

(iii) The number of subsets of the set X is $16 = 2^4 = 2^{\text{number of elements of } X}$.

We can generalise this for any finite set as follows.

> If the number of elements, or the cardinal number, of a finite set A is $n(A)$, the number of subsets of A is $2^{n(A)}$.

Proper subset

If the sets A and B are such that every element of A is an element of B but B has at least one element which is not an element of A then A is called a proper subset of B. This is expressed in symbols as $A \subset B$.

Examples (i) If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$ then every element of A is an element of B but the element '4' of B is not an element of A. So, $A \subset B$.

(ii) If P = {all the students of Delhi University}
and Q = {all the students in India}
then it is clear that $P \subset Q$.

> If A is a finite set containing $n(A)$ elements then the number of proper subsets of $A = 2^{n(A)} - 1$

Example The proper subsets of the set $A = \{1, 2\}$ are $\emptyset$, $\{1\}$ and $\{2\}$.
$A = \{1, 2\}$ itself is a subset of A but not a proper subset.
∴ the number of proper subsets of A is $2^2 - 1 = 4 - 1 = 3$.

Superset

Let A be a subset of B, i.e., $A \subseteq B$. Then, we say that B is a superset of A. We express this symbolically as $B \supseteq A$.

Example If $A = \{4, 6, 8\}$ and $B = \{2, 4, 6, 8, 10\}$ then $B \supseteq A$.

Equal sets

Two sets A and B are said to be equal if each element of A is an element of B and each element of B is an element of A. In other words, the sets A and B are equal if A is a subset of B and B is a subset of A, that is, if $A \subseteq B$ and $B \subseteq A$. The equal sets A and B are denoted by $A = B$.

Examples (i) If $P = \{1, 2, 3\}$ and $Q = \{x : 2x - 1 < 6$, where $x \in N\}$ then $P = Q$.

(ii) If $A = \{2, 4, 6\}$ and $B = \{6, 4, 2\}$ then $A = B$.

(iii) Let P = {letters of the word ROOF} and Q = {letters of the word FOR}. Then, P = {R, O, F} and Q = {F, O, R}. Thus, every element of P is a member of Q, or $P \subseteq Q$. Also, every element of Q is a member of P, or $Q \subseteq P$. So, $P = Q$.

> Equal sets are equivalent sets, but equivalent sets need not be equal sets.

Example If $P = \{x \mid x = 2n + 1$, where $n \in N$ and $n < 8\}$ and Q = {days in a week} then $n(P) = n(Q) = 7$. So, $P \leftrightarrow Q$.

However, Monday $\in Q$ but Monday $\notin P$. Therefore, $P \neq Q$.

Solved Examples

EXAMPLE 1 If P = {letters of the word BLAME}, Q = {letters of the word MEAL} and R = {letters of the word MALE} then identify the true and false statements.

(i) $P = Q$ (ii) $P \subseteq Q$ (iii) $Q \subseteq R$ (iv) $P \leftrightarrow Q$ (v) $Q \subseteq P$ (vi) $Q = R$

Solution The given sets can be represented in the tabular form as
$P = \{B, L, A, M, E\}$, $Q = \{M, E, A, L\}$ and $R = \{M, A, L, E\}$.

(i) False, because $B \in P$ but $B \notin Q$.

(ii) False, because the letter 'B' of the set P does not belong to Q.

(iii) True, because every element of Q is an element of R.

(iv) False, because $n(P) = 5$ and $n(Q) = 4$, and hence $n(P) \neq n(Q)$.

(v) True, because all the letters of the set Q are in P.

(vi) True, because $Q \subseteq R$ and $R \subseteq Q$.

EXAMPLE 2 If $A = \{x : x \in W \text{ and } 2x - 1 \leq 10\}$, $B = \{x : x = n^3$, where $n \in N$ and $n < 3\}$ and $C = \{x : x \in N \text{ and } 3x + 5 \leq 20\}$ then identify the true and false statements.

(i) $A \leftrightarrow C$ (ii) $A \subseteq B$ (iii) $B \subseteq C$ (iv) $C \subseteq A$

Solution In the tabular form, $A = \{0, 1, 2, 3, 4, 5\}$, $B = \{1, 8\}$ and $C = \{1, 2, 3, 4, 5\}$.

(i) False, because $n(A) = 6$ and $n(C) = 5$, and hence $n(A) \neq n(C)$.

(ii) False, because $3 \in A$ but $3 \notin B$.

(iii) False, because the element '8' of B is not an element of C.

(iv) True, because all the elements of C are elements of A.

EXAMPLE 3 If the universal set is $U = \{1, 2, 3, 4, 5\}$, and $A = \{x \mid x$ is a prime factor of 420$\}$ then write the set A in the tabular form. Also, write all the subsets of A which are singletons.

Solution $420 = 2 \times 2 \times 105 = 2 \times 2 \times 3 \times 5 \times 7$. So, the prime factors of 420 are 2, 3, 5 and 7.
But $7 \notin U$; so $A = \{2, 3, 5\}$.
Clearly, the singletons of A are $\{2\}$, $\{3\}$ and $\{5\}$.

Remember These

1. The set A is
   - (i) a finite set if $n(A)$ is finite
   - (ii) an infinite set if $n(A)$ is not finite
   - (iii) a singleton if $n(A) = 1$
   - (iv) the null set if $n(A) = 0$.
2. U denotes the universal set. It may be finite or infinite. Every set of a problem is a subset of the universal set of the problem.
3. Two sets A and B are equivalent if $n(A) = n(B)$.

4. If every element of a set A is an element of the set B, the set A is called a subset of the set B. This is denoted by $A \subseteq B$. For any set A, $A \subseteq A$ and $\emptyset \subseteq A$.
5. Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.
6. If A is a finite set containing $n(A)$ elements then the number of proper subsets of $A = 2^{n(A)} - 1$.

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EXERCISE 1B

1. State whether each of the following sets is a finite set or an infinite set.
   (i) The set of the multiples of 5
   (ii) The set of the positive integers greater than 20
   (iii) The set of the numbers which are factors of 32
   (iv) $A = \{x : x = 2n$, where $n \in N\}$
   (v) $B = \{x : x = 3n + 1$, where $n \in N$ and $n \le 20\}$
   (vi) The set of the positive integers which have the digit 2 in the unit's place
   (vii) $A = \left\{x : x = \frac{n-1}{n}\right.$, where $\left. n \in N\right\}$
2. Let the universal set be $U = \{x : x$ is a multiple of 3 and $0 < x < 37\}$, and $A = \{x : x$ is a multiple of 6$\}$. Write A by the roster method.
3. Let the universal set $U = \{1, 2, 3, 4, ..., 15\}$, $P = \{x : x = n^2, n \in N\}$ and $Q = \{x : x = 2^n, n \in W\}$. Write P and Q in the tabular form. Also, find the cardinal numbers of P and Q.
4. (i) Let R = {letters of the word APPLE} and S = {letters of the word PALE}. Find the cardinality of the sets R and S. Are R and S equivalent sets? Are R and S equal sets?
   (ii) Let B = {letters of the word WARD}, and C = {letters of the word DRAW}. Is $B \leftrightarrow C$? Is $B = C$?
5. State whether each of the following statements is true or false.
   (i) $\{2, 5\} \subseteq \{2, 15, 3, 5\}$
   (ii) $\{1, 2, 8\} \subseteq \{6, 2, 3, 1, 8\}$
   (iii) $\{0, 3, 5\} \subseteq \{x : x$ is a natural number$\}$
   (iv) $\{x : x$ is an even number$\} \subseteq \{x : x$ is an integer$\}$
   (v) $\{x : x$ is a multiple of 3$\} \subseteq \{x : x$ is an integer$\}$
   (vi) $\{p, q, r\} \subseteq \{x : x$ is a small letter of the English alphabet$\}$
   (vii) $\{x : x$ is a prime number between 10 and 20$\} \subseteq \{x : x$ is odd and $12 \le x < 25\}$
6. If $P = \{4, 5, 6, 7, 8, 9\}$, $Q = \{2, 3, 4\}$, $R = \{4, 5, 8\}$ then which of the following statements are true?
   (i) $P \subseteq Q$ (ii) $Q \subseteq R$ (iii) $R \subseteq P$ (iv) $P \subseteq R$
   (v) $P \leftrightarrow Q$ (vi) $Q \leftrightarrow R$ (vii) $Q = R$

7. Find the subsets of the following.
   (i) $A = \{4, 8\}$ (ii) $B = \{3, 6, 9\}$ (iii) $C = \{0, 3, 4, 5\}$
8. If P = {letters of the word ONE}, Q = {letters of the word NONE} and R = {letters of the word GONE} then which of the following statements are true and which are false?
   (i) $P \subset Q$ (ii) $Q \subset R$ (iii) $P \subseteq R$ (iv) $P \subset R$
   (v) $P = Q$ (vi) $Q = R$

ANSWERS

1. (i) Infinite set (ii) Infinite set (iii) Finite set (iv) Infinite set (v) Finite set (vi) Infinite set (vii) Infinite set
2. {6, 12, 18, 24, 30, 36}
3. $P = \{1, 4, 9\}$, $Q = \{1, 2, 4, 8\}$, $n(P) = 3$, $n(Q) = 4$
4. (i) $n(R) = 4$, $n(S) = 4$; R and S are equivalent sets; $R = S$ (ii) Yes, Yes
5. (i) True (ii) True (iii) False (iv) True (v) True (vi) True (vii) False
6. (iii) and (vi) are true
7. (i) ∅, {4}, {8} and {4, 8} (ii) ∅, {3}, {6}, {9}, {3, 6}, {3, 9}, {6, 9} and {3, 6, 9}
   (iii) ∅, {0}, {3}, {4}, {5}, {0, 3}, {0, 4}, {0, 5}, {3, 4}, {3, 5}, {4, 5}, {0, 3, 4}, {0, 3, 5}, {0, 4, 5}, {3, 4, 5} and {0, 3, 4, 5}
8. (i) False (ii) True (iii) True (iv) True (v) True (vi) False

❖

2

Operations and Venn Diagrams

Operations on Sets

When we carry out the operations of addition, multiplication, etc., on two numbers, we get new numbers. Similarly, when we carry out the operations of union and intersection on two sets, we get new sets. Just as we can find the difference between two numbers, we can also find the difference between two sets.

Union of two sets

The union of two sets A and B is the set of all the elements that belong to either A or B or both. It is denoted by $A \cup B$ (read as 'A union B').

We can write, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Examples (i) Let $A = \{2, 4, 6\}$ and $B = \{6, 8, 10\}$.

Then, $A \cup B = \{2, 4, 6\} \cup \{6, 8, 10\} = \{2, 4, 6, 8, 10\}$.

(ii) Let $P = \{a, b, c\}$ and $Q = \{x, y, z\}$.

Then, $P \cup Q = \{a, b, c, x, y, z\}$.

(iii) Let $A = \{5, 6, 8\}$.

Then, $A \cup A = \{5, 6, 8\} \cup \{5, 6, 8\} = \{5, 6, 8\} = A$.

Also, $A \cup \emptyset = \{5, 6, 8\} \cup \emptyset = \{5, 6, 8\} = A$.

(iv) Let $A = \{a, b, c\}$ and $B = \{a, b, c, d\}$.

Here, $A \subsetneq B$. Then, $A \cup B = \{a, b, c, d\} = B$.

Note The results we have arrived at in (iii) and (iv) are always true. In other words, they are laws of operations on sets.

Intersection of two sets

The intersection of two sets A and B is the set of all the elements which belong to both A and B. It is denoted by $A \cap B$ (read as 'A intersection B').

We can write, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

If A and B do not have any element in common then $A \cap B$ = null set = $\emptyset$.

Examples (i) Let $A = \{2, 4, 6, 8\}$ and $B = \{4, 8, 12\}$.

Then, $A \cap B = \{2, 4, 6, 8\} \cap \{4, 8, 12\} = \{4, 8\}$.

(ii) Let $P = \{1, 3, 5, 7, 9, 11, 13\}$ and $Q = \{2, 4, 6, 8, 10, 12, 14\}$.

Then, $P \cap Q = \{1, 3, 5, 7, 9, 11, 13\} \cap \{2, 4, 6, 8, 10, 12, 14\} = \emptyset$,

because there are no elements common to both P and Q.

(iii) Let $A = \{a, b, c\}$.

Then, $A \cap A = \{a, b, c\} \cap \{a, b, c\} = \{a, b, c\} = A$.

Also, $A \cap \emptyset = \{a, b, c\} \cap \emptyset = \emptyset$.

(iv) Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2\}$.

Here $B \subseteq A$. Then, $A \cap B = \{-2, -1, 0, 1, 2\} \cap \{0, 1, 2\} = \{0, 1, 2\} = B$.

Note (iii) and (iv) are also laws of operations on sets.

Disjoint sets

Two sets A and B are called disjoint sets if they have no elements in common, that is, if $A \cap B = \emptyset$.

Example Let $A = \{x : x$ is a positive integer$\}$ and

$B = \{x : x$ is a negative integer$\}$.

Then, $A = \{1, 2, 3, 4, \ldots\}$ and $B = \{-1, -2, -3, -4, \ldots\}$.

Then, $A \cap B = \{1, 2, 3, 4, \ldots\} \cap \{-1, -2, -3, -4, \ldots\} = \emptyset$.

So, A and B are disjoint sets.

Overlapping sets

Two sets A and B are called overlapping sets if they have at least one element in common, that is, if $A \cap B \neq \emptyset$.

Example Let $A = \{x : x$ is prime and $x < 10\}$ and

$B = \{$first two natural numbers$\}$.

Then, $A = \{2, 3, 5, 7\}$ and $B = \{1, 2\}$.

So, $A \cap B = \{2, 3, 5, 7\} \cap \{1, 2\} = \{2\}$.

$\therefore$ $A \cap B \neq \emptyset$. So, A and B are overlapping sets.

Difference of two sets

Let A and B be any two sets. Then, the difference of A and B is the set of elements which belong to A but not to B. This is denoted by $A - B$ or $A \setminus B$.

We can write $A - B = \{x : x \in A$ and $x \notin B\}$.

Similarly, $B - A = \{x : x \in B$ and $x \notin A\}$.

Example Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$.

Then, $A - B = \{1, 3\}$ and $B - A = \{6\}$.

Complement of a set

Given the universal set U, the complement (or complementary set) of a set A is the set of those elements of U which are not elements of A. It is denoted by A' (or A^C or $\bar{A}$). Symbolically, $A' = \{x : x \in U$ and $x \notin A\}$. In other words, A' is the difference of the universal set (U) and A. Mathematically,

$$A' = U - A.$$

Examples (i) Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$.

Then, $A' = U - A = \{1, 2, 3, 4, 5\} - \{1, 3, 5\} = \{2, 4\}$.

(ii) Let $U = \{x : x \in N$ and $x \le 10\}$ and $A = \{x : x \in W$ and $4 \le x \le 6\}$.
Then, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{4, 5, 6\}$.
Thus, $A' = U - A = \{1, 2, 3, 7, 8, 9, 10\}$.

Solved Examples

EXAMPLE 1 **Let A = {letters of the word CRICKET} and B = {letters of the word HOCKEY}. Find (i) $A \cup B$, (ii) $B \cup A$, (iii) $A \cap B$, (iv) $B \cap A$, (v) $A - B$, (vi) $B - A$.**

Solution In the tabular form, $A = \{C, R, I, K, E, T\}$ and $B = \{H, O, C, K, E, Y\}$.

(i) $A \cup B = \{C, R, I, K, E, T\} \cup \{H, O, C, K, E, Y\} = \{C, R, I, K, E, T, H, O, Y\}$.

(ii) $B \cup A = \{H, O, C, K, E, Y\} \cup \{C, R, I, K, E, T\} = \{H, O, C, K, E, Y, R, I, T\}$.

(iii) $A \cap B = \{C, R, I, K, E, T\} \cap \{H, O, C, K, E, Y\} = \{C, K, E\}$.

(iv) $B \cap A = \{H, O, C, K, E, Y\} \cap \{C, R, I, K, E, T\} = \{C, K, E\}$.

(v) $A - B = \{C, R, I, K, E, T\} - \{H, O, C, K, E, Y\} = \{R, I, T\}$.

(vi) $B - A = \{H, O, C, K, E, Y\} - \{C, R, I, K, E, T\} = \{H, O, Y\}$.

Note From (i) and (ii) it should be clear that $A \cup B = B \cup A$. Similarly, from (iii) and (iv), $A \cap B = B \cap A$. These two are laws of operations on sets.

EXAMPLE 2 **If $U = \{x : x \in W$ and $6 \le x \le 11\}$, $A = \{6, 8, 9\}$, $B = \{7, 8, 11\}$ and $C = \{6\}$, find the following sets.**

(i) A' (ii) B' (iii) C' (iv) $A - B$ (v) $(B \cup C)'$ (vi) $(A \cap C)'$
(vii) $A - (B \cup C)$ (viii) $A - (B \cap C)$

Solution Here, $U = \{x : x \in W$ and $6 \le x \le 11\} = \{6, 7, 8, 9, 10, 11\}$, $A = \{6, 8, 9\}$, $B = \{7, 8, 11\}$ and $C = \{6\}$.

(i) $A' = U - A = \{6, 7, 8, 9, 10, 11\} - \{6, 8, 9\} = \{7, 10, 11\}$.

(ii) $B' = U - B = \{6, 7, 8, 9, 10, 11\} - \{7, 8, 11\} = \{6, 9, 10\}$.

(iii) $C' = U - C = \{6, 7, 8, 9, 10, 11\} - \{6\} = \{7, 8, 9, 10, 11\}$.

(iv) $A - B = \{6, 8, 9\} - \{7, 8, 11\} = \{6, 9\}$.

(v) $B \cup C = \{7, 8, 11\} \cup \{6\} = \{6, 7, 8, 11\}$.
$\therefore$ $(B \cup C)' = U - (B \cup C) = \{6, 7, 8, 9, 10, 11\} - \{6, 7, 8, 11\} = \{9, 10\}$.

(vi) $A \cap C = \{6, 8, 9\} \cap \{6\} = \{6\}$.
$\therefore$ $(A \cap C)' = U - (A \cap C) = \{6, 7, 8, 9, 10, 11\} - \{6\} = \{7, 8, 9, 10, 11\}$.

(vii) $A - (B \cup C) = \{6, 8, 9\} - \{6, 7, 8, 11\} = \{9\}$.

(viii) $B \cap C = \{7, 8, 11\} \cap \{6\} = \emptyset$. So, $A - (B \cap C) = \{6, 8, 9\} - \{\emptyset\} = \{6, 8, 9\}$.

EXAMPLE 3 **Let $U = \{x : x \in N$ and $2 \le x \le 9\}$, $A = \{x : x$ is an even number$\}$, $B = \{x : x$ is a multiple of 3$\}$ and $C = \{x : x$ is a multiple of 4$\}$. Verify the following.**

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution Here, $U = \{2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$, $B = \{3, 6, 9\}$ and $C = \{4, 8\}$.

(i) $A \cup B = \{2, 4, 6, 8\} \cup \{3, 6, 9\} = \{2, 3, 4, 6, 8, 9\}$. So, $(A \cup B)' = \{5, 7\}$.
$A' = U - A = \{2, 3, 4, 5, 6, 7, 8, 9\} - \{2, 4, 6, 8\} = \{3, 5, 7, 9\}$.
$B' = U - B = \{2, 3, 4, 5, 6, 7, 8, 9\} - \{3, 6, 9\} = \{2, 4, 5, 7, 8\}$.
$\therefore \quad A' \cap B' = \{3, 5, 7, 9\} \cap \{2, 4, 5, 7, 8\} = \{5, 7\}$.

$\therefore \ (A \cup B)' = A' \cap B'$.

(ii) $A \cap B = \{6\}$; so $(A \cap B)' = \{2, 3, 4, 5, 7, 8, 9\}$.
$A' \cup B' = \{3, 5, 7, 9\} \cup \{2, 4, 5, 7, 8\} = \{2, 3, 4, 5, 7, 8, 9\}$.

$\therefore \ (A \cap B)' = A' \cup B'$.

(iii) $B \cap C = \emptyset$; so $A \cup (B \cap C) = A \cup \emptyset = A = \{2, 4, 6, 8\}$.
$(A \cup B) \cap (A \cup C) = \{2, 3, 4, 6, 8, 9\} \cap \{2, 4, 6, 8\} = \{2, 4, 6, 8\}$.

$\therefore \ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(iv) $A \cap (B \cup C) = \{2, 4, 6, 8\} \cap \{3, 4, 6, 8, 9\} = \{4, 6, 8\}$.
$(A \cap B) \cup (A \cap C) = \{6\} \cup \{4, 8\} = \{4, 6, 8\}$.

$\therefore \ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Note (i), (ii), (iii) and (iv) are laws of operations on sets.

Remember These

1. ***Symbol*** — ***Meaning***

Symbol	Meaning
$A \cup B$	The union of the sets A and B
$A \cap B$	The intersection of the sets A and B
$A - B$, or $A \setminus B$	The difference of the sets A and B
A', or A^C, or $\overline{A}$	The complementary set, or complement, of the set A

2. Two sets A and B are
   (i) disjoint if $A \cap B = \emptyset$ (ii) overlapping if $A \cap B \neq \emptyset$.
3. To find $A - B$, list all the members of A which are not members of B.
4. To find A', list all the members of the universal set U which are not members of A.
   For any set A, we can write $A \subseteq A$ and $\emptyset \subseteq A$.
5. ***Laws of operations on sets***
   (i) Idempotent laws: $A \cup A = A$, $A \cap A = A$, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$
   (ii) Commutative laws: $A \cup B = B \cup A$, $A \cap B = B \cap A$
   (iii) Associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
   (iv) Distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (v) De Morgan's laws: $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$
   (vi) $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$, $A \cap B \subseteq B$
   (vii) If $A \subseteq B$ then $A \cup B = B$, $A \cap B = A$ and $A - B = \emptyset$

EXERCISE 2A

1. If $A = \{3, 4, 5\}$ and $B = \{5, 6, 7, 8\}$ then find (i) $A \cup B$, (ii) $A \cap B$, (iii) $A - B$ and (iv) $B - A$, and also verify the following.
   (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$ (c) $A \cup A = A$ (d) $A \cup \emptyset = A$
   (e) $A \cap A = A$ (f) $A \cap \emptyset = \emptyset$
2. Let $U = \{5, 6, 7, 8, 9, 10\}$, $P = \{5, 6, 7, 9\}$, $Q = \{5, 6, 10\}$ and $R = \{5, 9\}$. Find the following
   (i) P' (ii) Q' (iii) R' (iv) $P - Q$
   (v) $Q \cup R$ (vi) $P \cap R$ (vii) $P - (Q \cup R)$ (viii) $P - (Q \cap R)$
3. If A = {letters of the word RICE}, B = {letters of the word NICE}, find the following
   (i) $A \cup B$ (ii) $B \cup A$ (iii) $A \cap B$ (iv) $B \cap A$
   (v) $A - B$ (vi) $B - A$
   Also, verify that
   (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$ (c) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
4. Let the universal set be $U = \{x : x \in W \text{ and } x < 9\}$. Also, A = {factors of 6} and B = {factors of 8}. Find the following sets.
   (i) A' (ii) B' (iii) $A' \cap B'$ (iv) $A' \cup B'$
   Also, verify the following.
   (a) $(A \cup B)' = A' \cap B'$ (b) $(A \cap B)' = A' \cup B'$ (c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. If $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$ and $B = \{1, 3, 5, 7\}$, verify the following.
   (i) $A' = B$ (ii) $B' = A$ (iii) $A' \cup B' = U$ (iv) $A \cup B = U$

ANSWERS

1. (i) {3, 4, 5, 6, 7, 8} (ii) {5} (iii) {3, 4} (iv) {6, 7, 8}
2. (i) {8, 10} (ii) {7, 8, 9} (iii) {6, 7, 8, 10} (iv) {7, 9} (v) {5, 6, 9, 10} (vi) {5, 9} (vii) {7} (viii) {6, 7, 9}
3. (i) {R, I, C, E, N} (ii) {R, I, C, E, N} (iii) {I, C, E} (iv) {I, C, E} (v) {R} (vi) {N}
4. (i) {0, 4, 5, 7, 8} (ii) {0, 3, 5, 6, 7} (iii) {0, 5, 7} (iv) {0, 3, 4, 5, 6, 7, 8}

Venn Diagrams

A Venn diagram is a closed geometrical figure used to denote a set or a combination of sets within a given universal set. Points within such a closed figure represent the elements of the set.

The universal set (U) is usually represented by a rectangle. All other sets are represented by closed curves (usually circles or ovals) within the rectangle. All the elements of a set are indicated by points inside the circle representing the set. The elements can also be written inside the circle without marking the points. An object that is not a member of a set is placed outside the circle representing the set.

Example The set $A = \{2, 3, 4, 5\}$ can be represented by either of the two Venn diagrams shown. Here, $6 \notin A$.

Relationships between sets

Venn diagrams can be used to represent the relationships between sets.

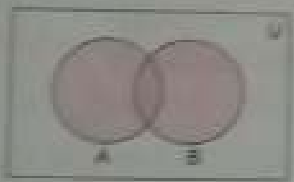
Subset of a set

If $A \subseteq B$, the set A is represented by a circle within the circle representing the set B.

Union of sets

The set $A \cup B$ may be represented by the shaded portion of each of the following Venn diagrams under three situations—(i) A and B are overlapping sets, (ii) A and B are disjoint sets, and (iii) A is a subset of B (i.e., $A \subseteq B$).

(i)



A and B are overlapping sets.

(ii)

A and B are disjoint sets.

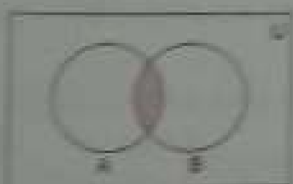
(iii)

$A \subseteq B$

Intersection of sets

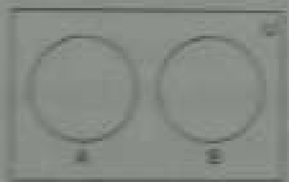
The shaded portion of each of the following Venn diagrams represents the set $A \cap B$ when (i) A and B are overlapping sets, (ii) A and B are disjoint sets, and (iii) $A \subseteq B$. No portion of the diagram is shaded when A and B are disjoint sets, because in this case, $A \cap B = \varnothing$.

(i)



A and B are overlapping sets.

(ii)



A and B are disjoint sets.

(iii)

$A \subseteq B$

Difference of two sets

The shaded portions of the following Venn diagrams represent the difference set $A - B$ when (i) A and B are overlapping sets, (ii) A and B are disjoint sets, and (iii) $A \subseteq B$. No portion of the third diagram is shaded because $A - B = \varnothing$.

(i)

A and B are overlapping sets

(ii)

A and B are disjoint sets

(iii)

A is a subset of B

The shaded portions of the following diagrams show the difference set $B - A$ in the similar three cases.

(i)

A and B are overlapping sets

(ii)

A and B are disjoint sets

(iii)

A is a subset of B

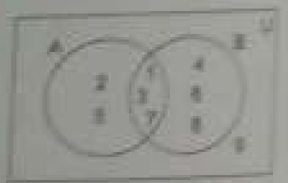
Complement of a set

The shaded portion of the adjoining Venn diagram represents the set A', which is the complement of the set A.

A': shaded portion

Interpreting a Venn diagram

In the Venn diagram shown alongside, the rectangle representing the universal set U contains the elements 2, 5, 1, 3, 7, 4, 6, 8 and 9.



$\therefore \quad U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $n(U) = 9$.

The closed region in the rectangle representing the set A contains the elements 2, 5, 1, 3 and 7, while that representing the set B contains 1, 3, 7, 4, 6 and 8. Thus,

$A = \{1, 2, 3, 5, 7\}$ and $n(A) = 5$;

$B = \{1, 3, 7, 4, 6, 8\} = \{1, 3, 4, 6, 7, 8\}$ and $n(B) = 6$;

$A \cup B = \{2, 5, 1, 3, 7, 4, 6, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $n(A \cup B) = 8$;

$A \cap B = \{1, 3, 7\}$ and $n(A \cap B) = 3$;

$A - B = \{2, 5\}$ and $n(A - B) = 2$;

$B - A = \{4, 6, 8\}$ and $n(B - A) = 3$;

$A' = \{4, 6, 8, 9\}$ and $n(A') = 4$;

$B' = \{2, 5, 9\}$ and $n(B') = 3$;

$(A \cup B)' = \{9\}$ and $n(A \cup B)' = 1$;

$(A \cap B)' = \{2, 5, 4, 6, 8, 9\}$ and $n(A \cap B)' = 6$.

From these, we can arrive at certain general relations between the cardinal numbers of sets, which are known as the **cardinal properties of sets**.

1. $n(A) + n(B) = 5 + 6 = 11$, $n(A \cup B) = 8$ and $n(A \cap B) = 3$. Therefore,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

You know that $n(A \cap B) = 0$ when A and B are disjoint sets. Therefore, for disjoint sets,

$$n(A \cup B) = n(A) + n(B).$$

2. $n(A) = 5$, $n(A') = 4$ and $n(U) = 9$. Therefore,

$$n(A) + n(A') = n(U).$$

3. $n(A \cup B) = 8$, $n(B) = 6$ and $n(A - B) = 2$. Therefore,

$$n(A - B) + n(B) = n(A \cup B).$$

4. $n(A \cap B) = 3$, $n(A) = 5$ and $n(A - B) = 2$. Therefore,

$$n(A) - n(A - B) = n(A \cap B).$$

5. $n(A - B) = 2$, $n(B - A) = 3$, $n(A \cup B) = 8$ and $n(A \cap B) = 3$. Therefore,

$$n(A - B) + n(B - A) = n(A \cup B) - n(A \cap B).$$

Solved Examples

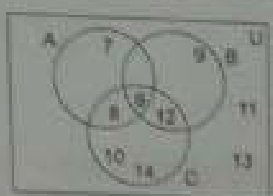
EXAMPLE 1 **Let $U = \{x : x \in W$ and $6 \le x < 15\}$, $A = \{x : x$ is a positive integer and $x \le 8\}$, $B = \{x : x$ is a multiple of 3$\}$ and $C = \{$even integers$\}$.**

Represent the given sets by a Venn diagram and then find the following.

(i) $A \cap B$ (ii) $B \cap C$ (iii) $C \cap A$ (iv) A' (v) B' (vi) C' (vii) $(A \cup B)'$ (viii) $(B \cup C)'$

Solution We can represent the given sets in the tabular form as $U = \{6, 7, 8, 9, \ldots, 14\}$, $A = \{6, 7, 8\}$, $B = \{6, 9, 12\}$ and $C = \{6, 8, 10, 12, 14\}$.

We can arrive at the following from the adjoining Venn diagram which represents the given sets.



(i) $A \cap B = \{6\}$ (ii) $B \cap C = \{6, 12\}$

(iii) $C \cap A = \{6, 8\}$ (iv) $A' = \{9, 10, 11, 12, 13, 14\}$

(v) $B' = \{7, 8, 10, 11, 13, 14\}$ (vi) $C' = \{7, 9, 11, 13\}$

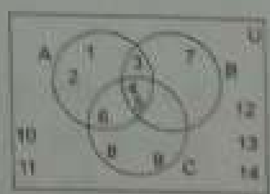
(vii) $(A \cup B)' = \{10, 11, 13, 14\}$ (viii) $(B \cup C)' = \{7, 11, 13\}$

EXAMPLE 2 **Find the following sets from the adjoining Venn diagram.**

(i) U (ii) A (iii) B (iv) C (v) A'

(vi) B' (vii) C' (viii) $A \cap B$ (ix) $B \cap C$ (x) $A \cap C$

(xi) $A - B$ (xii) $B - C$ (xiii) $A - C$ (xiv) $(A \cup B)'$



Solution (i) $U = \{1, 2, 3, 4, \ldots, 14\}$ (ii) $A = \{1, 2, 3, 4, 5, 6\}$

(iii) $B = \{3, 4, 5, 7\}$ (iv) $C = \{4, 5, 6, 8, 9\}$

(v) $A' = \{7, 8, 9, \ldots, 14\}$ (vi) $B' = \{1, 2, 6, 8, 9, 10, 11, 12, 13, 14\}$

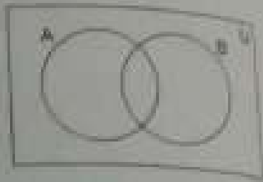
(vii) $C' = \{1, 2, 3, 7, 10, 11, 12, 13, 14\}$ (viii) $A \cap B = \{3, 4, 5\}$

(ix) $B \cap C = \{4, 5\}$ (x) $A \cap C = \{4, 5, 6\}$

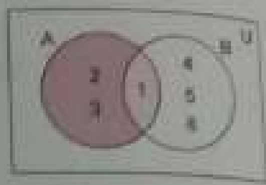
(xi) $A - B = \{1, 2, 6\}$ (xii) $B - C = \{3, 7\}$

(xiii) $A - C = \{1, 2, 3\}$ (xiv) $(A \cup B)' = \{8, 9, 10, 11, 12, 13, 14\}$

EXAMPLE 3 **If $A = \{1, 2, 3\}$, $A \cap B = \{1\}$ and $B - A = \{4, 5, 6\}$ then fill the elements in the Venn diagram. Also, tabulate the elements of the sets B, $A - B$ and $A \cup B$.**



Solution Let us first fill the elements of $A \cap B$ and $B - A$. Then, let us fill the elements of A that are not contained in $A \cap B$. These elements are 2 and 3. Now, from the Venn diagram, $B = \{1, 4, 5, 6\}$, $A - B = \{2, 3\}$ and $A \cup B = \{2, 3, 1, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$.



EXAMPLE 4 **Let U be the universal set, and P and Q be any two sets. If $n(U) = 60$, $n(P) = 30$, $n(Q) = 20$ and $n(P \cap Q)' = 50$, find the following.**

(i) $n(P \cap Q)$ (ii) $n(P \cup Q)$ (iii) $n(P - Q)$ (iv) $n(Q - P)$

Solution (i) We know that $n(A) + n(A') = n(U)$.

Replacing A by $P \cap Q$, we obtain $n(P \cap Q) + n(P \cap Q)' = n(U)$.

$\therefore$ $n(P \cap Q) + 50 = 60$ or $n(P \cap Q) = 10$.

(ii) We know that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

$\therefore$ $n(P \cup Q) = 30 + 20 - 10 = 40$.

(iii) $\because$ $n(P - Q) + n(Q) = n(P \cup Q)$.

$\therefore$ $n(P - Q) = n(P \cup Q) - n(Q) = 40 - 20 = 20$.

(iv) $\because$ $n(P - Q) + n(Q - P) = n(P \cup Q) - n(P \cap Q)$ and $20 + n(Q - P) = 40 - 10 = 30$.

$\therefore$ $n(Q - P) = 30 - 20 = 10$.

EXAMPLE 5 **Let U be the universal set, and A and B any two sets. If $n(U) = 10$, $n(A) = 4$, $n(B) = 3$ and $n(A \cup B)' = 4$, find the following.**

(i) $n(A')$ (ii) $n(B')$ (iii) $n(A \cup B)$ (iv) $n(A \cap B)$ (v) $n(A - B)$ (vi) $n(B - A)$

Solution (i) $n(A) + n(A') = n(U)$. So, $n(A') = n(U) - n(A) = 10 - 4 = 6$.

(ii) $n(B) + n(B') = n(U)$. So, $n(B') = n(U) - n(B) = 10 - 3 = 7$.

(iii) $n(A) + n(A') = n(U)$. Replacing A by $A \cup B$,

$n(A \cup B) + n(A \cup B)' = n(U)$ or $n(A \cup B) + 4 = 10$.

$\therefore$ $n(A \cup B) = 10 - 4 = 6$.

(iv) $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 4 + 3 - 6 = 1$.

(v) $n(A - B) + n(B) = n(A \cup B)$. So, $n(A - B) + 3 = 6$ or $n(A - B) = 3$.

(vi) $n(A - B) + n(B - A) = n(A \cup B) - n(A \cap B)$.

So, $3 + n(B - A) = 6 - 1$ or $n(B - A) = 2$.

EXAMPLE 6 **Let A and B be any two sets. If $n(A - B) = 20$, $n(A \cup B) = 60$, $n(A \cap B) = 10$, find the following.**

(i) $n(B - A)$ (ii) $n(A)$ (iii) $n(B)$

Solution (i) $n(A-B)+n(B-A)=n(A\cup B)-n(A\cap B)$.

$\therefore\quad 20+n(B-A)=60-10=50$ or $n(B-A)=50-20=30$.

(ii) $n(A)-n(A-B)=n(A\cap B)$. So, $n(A)-20=10$ or $n(A)=30$.

(iii) $n(A-B)+n(B)=n(A\cup B)$. So, $20+n(B)=60$ or $n(B)=60-20=40$.

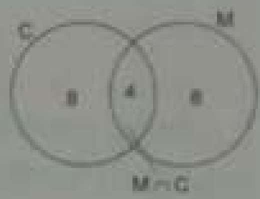
EXAMPLE 7 **Out of 20 students of a class who like either cocoa or milk or both, 12 like cocoa, while 4 like both. Draw a Venn diagram and find the number of students who like (i) milk, (ii) only milk, (iii) only cocoa.**

Solution Let C = {students who like cocoa} and M = {students who like milk}.

Then, $n(C)$ = (number of students who like cocoa) = 12,

$n(C\cup M)$ = (number of students who like either cocoa or milk or both) = 20

and $n(C\cap M)$ = (number of students who like both) = 4.



Now, $n(C\cup M)=n(M)+n(C)-n(C\cap M)$

or $20=n(M)+12-4$ or $n(M)=12$.

$\therefore$ the number of students who like milk is 12.

But the number of students who like both milk and cocoa is 4.

$\therefore$ the number of students who like only milk is $n(M)-n(C\cap M)=12-4=8$.

Also, the number of students who like only cocoa is $n(C)-n(C\cap M)=12-4=8$.

EXAMPLE 8 **Out of a class of 100 students, 70 watch cartoons, 80 watch sports, and all watch either cartoons or sports or both. Find the number of students who watch (i) both cartoons and sports, (ii) only cartoons, (iii) only sports.**

Solution Let C = {students who watch cartoons}, S = {students who watch sports},

$C\cup S$ = {students who watch either cartoons or sports or both} and

$C\cap S$ = {students who watch both}.

Given, $n(C)=70$, $n(S)=80$ and $n(C\cup S)=100$.

Let $n(C\cap S)$ = (number of students who watch both)

$=y$. (See Venn diagram.)

Then, the number of students who watch only cartoons $=70-y$

and the number of students who watch only sports $=80-y$.

$\therefore$ total number of students $=100=(70-y)+y+(80-y)$.

$\therefore\quad 100=150-y$ or $y=50$.

(i) So, the number of students who watch both is $y=50$.

(ii) The number of students who watch only cartoons is $(70-y)=70-50=20$.

(iii) The number of students who watch only sports is $(80-y)=80-50=30$.

EXAMPLE 9 **Out of a group of 500 people, 240 watch cricket, 50 watch football, and 15 watch both. Find the number of people who watch**

(i) only cricket **(ii) only football**

(iii) either cricket or football or both **(iv) neither cricket nor football.**

Solution Let U = {the entire group}, C = {people who watch cricket} and F = {people who watch football}.

Given, $n(C) = 240$, $n(F) = 50$ and $n(C \cap F) = 15$.

(i) The number of people who watch only cricket is
$n(C) - n(C \cap F) = 240 - 15 = 225.$

(ii) The number of people who watch only football is
$n(F) - n(C \cap F) = 50 - 15 = 35.$

(iii) The number of people who watch either cricket or football or both is
$n(C \cup F) = n(C) + n(F) - n(C \cap F) = 240 + 50 - 15 = 275.$

(iv) The number of people who watch neither cricket nor football is
$n(U) - n(C \cup F) = 500 - 275 = 225.$

Remember These

Some cardinal properties of sets

(i) $n(A) + n(A') = n(U)$

(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(iii) $n(A \cup B) = n(A - B) + n(B)$

(iv) $n(A) - n(A - B) = n(A \cap B)$

(v) $n(A - B) + n(B - A) + n(A \cap B) = n(A \cup B)$

EXERCISE 2B

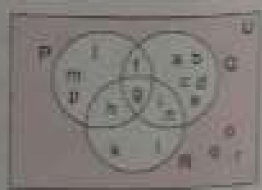
1. Let the universal set be $U = \{0, 1, 2, 3, \ldots, 9\}$, and $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 3, 5, 8\}$ and $C = \{2, 5, 6, 7\}$. Draw a Venn diagram to represent these sets. Also, find the following.

(i) $A \cap B$ (ii) $B \cap C$ (iii) $C \cap A$ (iv) $A \cup B$
(v) $B \cup C$ (vi) $C \cup A$ (vii) A' (viii) B'
(ix) C' (x) $A - B$ (xi) $B - C$ (xii) $(A \cup B)'$

2. Use the given Venn diagram to find the following sets.

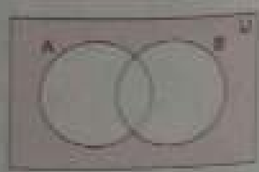
(i) U (ii) A (iii) B
(iv) $A \cup B$ (v) A' (vi) B'
(vii) $A \cap B$

3. Use the given Venn diagram to find the following.



(i) U (ii) P (iii) Q
(iv) R (v) P' (vi) Q'
(vii) R' (viii) $P \cap Q$ (ix) $Q \cap R$
(x) $P - Q$ (xi) $Q - R$ (xii) $P - R$

4. If $A = \{a, b, c\}$, $B - A = \{d, e, f, g\}$ and $A \cap B = \{b, c\}$ then fill the elements of the given sets in the Venn diagram, and tabulate the elements of the sets B, $A - B$ and $A \cup B$.



5. If $A - B = \{1, 3, 7, 8\}$, $B - A = \{2, 4, 6\}$ and $A \cap B = \{5\}$ then use a Venn diagram to display the elements of the given sets, and tabulate the elements of the sets A, B and $A \cup B$.

6. (a) If $n(A) = 10$, $n(B) = 15$ and $n(A \cap B) = 7$ then find $n(A \cup B)$.
   (b) If $n(P) = 50$, $n(Q) = 30$ and $n(P \cup Q) = 70$ then find $n(P \cap Q)$.

7. Let U be the universal set, and A and B be any two sets. If $n(U) = 20$, $n(A) = 12$, $n(B) = 4$ and $n(A \cap B)' = 17$, find the following.
   (i) $n(A \cap B)$ (ii) $n(A \cup B)$ (iii) $n(A - B)$ (iv) $n(B - A)$

8. Let U denote the universal set, and A and B be any two sets. If $n(U) = 25$, $n(A) = 17$, $n(B) = 7$ and $n(A \cup B)' = 2$, find the following.
   (i) $n(A')$ (ii) $n(B')$ (iii) $n(A \cup B)$ (iv) $n(A \cap B)$
   (v) $n(A - B)$ (vi) $n(B - A)$

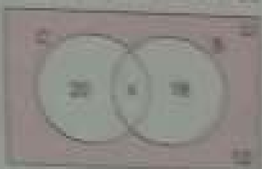
9. If A and B be any two sets, where $n(A - B) = 8$, $n(A \cup B) = 20$ and $n(A \cap B) = 6$, find the following.
   (i) $n(B - A)$ (ii) $n(A)$ (iii) $n(B)$

10. 50 students of a class like either oranges or grapes or both. 20 of them like oranges, while 10 like both. Draw a Venn diagram and find how many like (i) only oranges, (ii) grapes, (iii) only grapes?

11. A class has 100 students. If 60 know English, 80 know Hindi, and all the students of the class know either Hindi or English or both, find the number of students who know both the languages.

12. Out of a group of 1000 people, 560 have a TV, 250 have a computer and 130 have both. Find the number of people who have
    (i) only a TV (ii) only a computer
    (iii) either a TV or a computer or both (iv) neither a TV nor a computer.

13. Let U = {students in a sports class}, C = {students who like cricket} and S = {students who like soccer}. There are 60 students in the class. Use the given Venn diagram to answer the following.
    (i) What is the value of x?
    (ii) How many students like cricket?
    (iii) How many do not like soccer?



ANSWERS

1. (i) {2, 8} (ii) {2, 5} (iii) {2, 6} (iv) {1, 2, 3, 4, 5, 6, 8}
   (v) {2, 3, 5, 6, 7, 8} (vi) {1, 2, 4, 5, 6, 7, 8} (vii) {0, 3, 5, 7, 9}
   (viii) {0, 1, 4, 6, 7, 9} (ix) {0, 1, 3, 4, 8, 9} (x) {1, 4, 6}
   (xi) {3, 8} (xii) {0, 7, 9}

2. (i) {2, 4, 6, 8, 10, 12, 14, 16, 18, 20} (ii) {2, 8, 10, 12} (iii) {4, 10, 12, 14, 16}
   (iv) {2, 4, 8, 10, 12, 14, 16} (v) {4, 6, 14, 16, 18, 20} (vi) {2, 6, 8, 18, 20} (vii) {10, 12}

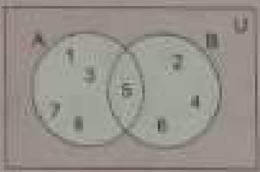
3. (i) {a, b, c, d, e, f, g, h, i, ..., r} (ii) {f, g, h, j, m, p} (iii) {a, b, c, d, e, f, g, i, n} (iv) {g, h, i, k, l, n}
   (v) {a, b, c, d, e, i, k, l, n, o, q, r} (vi) {h, j, k, l, m, o, p, q, r} (vii) {a, b, c, d, e, f, j, m, o, p, q, r}
   (viii) {f, g} (ix) {g, i, n} (x) {h, j, m, p} (xi) {a, b, c, d, e, f} (xii) {f, j, m, p}

4. 

$B = \{b, c, d, e, f, g\}$

$A - B = \{a\}$

$A \cup B = \{a, b, c, d, e, f, g\}$

5. 

$A = \{1, 3, 5, 7, 8\}$

$B = \{2, 4, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

6. (a) 18 (b) 10

7. (i) 3 (ii) 13 (iii) 9 (iv) 1

8. (i) 8 (ii) 18 (iii) 23 (iv) 1 (v) 16 (vi) 6

9. (i) 6 (ii) 14 (iii) 12

10. 

(i) 10 (ii) 40 (iii) 30

11. 40 students

12. (i) 430 (ii) 120 (iii) 680 (iv) 320

13. (i) 12 (ii) 32 (iii) 32

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Revision Exercise

1. Which of the following collections are sets?
   (i) The collection of the capitals of the states of India
   (ii) The collection of the top singers of the Indian film industry
   (iii) The collection of the natural numbers which are perfect cubes and are less than 100
   (iv) The collection of good politicians of India
2. Write each of the following sets by the roster method.
   (i) The set of the months of a year having 30 days
   (ii) The set of the equal sides of an equilateral triangle *ABC*
   (iii) The set of the planets of the solar system
   (iv) The set of the months of a year that end in '-ber'
   (v) The set of the colours used in traffic lights
   (vi) The set of the oceans of the world
3. Express the following sets in the set-builder form.
   (i) {4, 8, 12, 16, 20} (ii) {–15, –10, –5, 0, 5, 10, 15, 20}
   (iii) $\left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\}$
4. If L = {letters of the word RATE}, M = {letters of the word TEAR}, N = {letters of the word RARE} and O = {letters of the word TREAT} then which of the following statements are true and which are false?
   (i) $L \subset M$ (ii) $N \subset M$ (iii) $N \subset O$ (iv) $L = O$
   (v) $L = N$ (vi) $M = O$
5. Let G = {letters of the word EUROPE}, H = {letters of the word NORWAY} and I = {letters of the word ENGLAND}. Find the following.
   (i) $G \cup H$ (ii) $H \cup I$ (iii) $G \cup I$ (iv) $G \cap H$
   (v) $H \cap I$ (vi) $G \cap I$ (vii) $G - H$ (viii) $H - I$
   (ix) $G - I$ (x) $G - (H \cap I)$ (xi) $H - (G \cap I)$ (xii) $I - (G \cap H)$
6. Let the universal set be $U = \{3, 4, 5, 6, 7, 8, 9, 10\}$, and $A = \{4, 5, 9, 10\}$, $B = \{3, 5, 9\}$ and $C = \{3, 10\}$. Find the following.
   (i) A' (ii) B' (iii) C' (iv) $A - B$
   (v) $C - A$ (vi) $A \cap C$ (vii) $C - (A \cap B)$ (viii) $A - (B \cup C)$
7. Let the universal set be $U = \{x : x \in N \text{ and } x \leq 18\}$, and A = {factors of 12} and B = {factors of 18}. Find the following.
   (i) A' (ii) B' (iii) $A' \cup B'$ (iv) $A' \cap B'$.
   Also, verify that (a) $(A \cup B)' = A' \cap B'$ and (b) $(A \cap B)' = A' \cup B'$.
8. Let the universal set be $U = \{x : x \in W \text{ and } x \leq 10\}$, and A = {prime numbers less than 10}, B = {even numbers less than 10} and C = {factors of 10}. Verify the following.
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
   (iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. Let the universal set be U = {letters of the English alphabet up to the letter 'm'}, and A = {a, b, c}, B = {a, b, c, d, e, f} and C = {a, e, f, g, h, i, j}. Represent the given sets by a Venn diagram and find the following.
   (i) $A \cap B$ (ii) $B \cap C$ (iii) $C \cap A$ (iv) A'

(v) B' (vi) C' (vii) $(A \cap B)'$ (viii) $(B \cap C)'$

(ix) $(A \cup B) \cap C$ (x) $(A \cap B) \cup C$

10. In a class, 75 students play either football or cricket or both. Of these, 35 play football, while 20 play both. Draw a Venn diagram and find how many play (i) only cricket and (ii) only football.

11. Out of a group of 56 students, 36 opted for mathematics, and 45 opted for biology. How many students opted for (i) both subjects, (ii) only mathematics and (iii) only biology? Draw a Venn diagram to represent the sets.

ANSWERS

1. (i), (iii)

2. (i) {April, June, September, November} (ii) {AB, BC, CA}
(iii) {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}
(iv) {September, October, November, December} (v) {red, yellow, green}
(vi) {Indian, Atlantic, Pacific, Arctic, Antarctic}

3. (i) $\{x : x = 4n$, where $x \in N$ and $n \leq 6\}$ (ii) $\{x \mid x = 5n$, where $n \in I$ and $-3 \leq n \leq 4\}$
(iii) $\left\{x : x = \frac{n+1}{n}, \text{ where } x \in N\right\}$

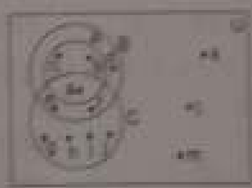
4. (i) False (ii) True (iii) True (iv) True (v) False (vi) True

5. (i) {E, U, R, O, P, N, W, A, Y} (ii) {N, O, R, W, A, Y, E, G, L, D} (iii) {E, U, R, O, P, N, G, L, A, D}
(iv) {R, O} (v) {N, A} (vi) {E} (vii) {E, U, P} (viii) {O, R, W, Y} (ix) {U, R, O, P} (x) {E, U, R, O, P}
(xi) {N, O, R, W, A, Y} (xii) {E, N, G, L, A, D}

6. (i) {3, 6, 7, 8} (ii) {4, 6, 7, 8, 10} (iii) {4, 5, 6, 7, 8, 9} (iv) {4, 10} (v) {3} (vi) {10} (vii) {3, 10}
(viii) {4}

7. (i) {5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18} (ii) {4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17}
(iii) {4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18} (iv) {5, 7, 8, 10, 11, 13, 14, 15, 16, 17}

9. (i) {a, b, c} (ii) {a, e, f} (iii) {a} (iv) {d, e, f, g, h, i, j, k, l, m} (v) {g, h, i, j, k, l, m} (vi) {b, c, d, k, l, m}
(vii) {d, e, f, g, h, i, j, k, l, m} (viii) {b, c, d, g, h, i, j, k, l, m} (ix) {a, e, f} (x) {a, b, c, e, f, g, h, i, j}



10.

(i) 40 (ii) 15

11. (i) 25 (ii) 11 (iii) 20

❖

1

Rational Numbers

Any number which can be expressed in the form of $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is called a rational number or simply a rational.

The set of rational numbers is denoted by Q.

$\therefore \quad Q = \left\{\frac{p}{q} : p \in I, q \in I \text{ and } q \neq 0\right\}$.

Thus, $\frac{5}{7}, \frac{-7}{11}, \frac{8}{-13}$ and $\frac{-12}{-17}$ are some rational numbers.

Examples (i) All fractions are rational numbers.

$\frac{2}{7}, \frac{3}{13}, \frac{-17}{65}$ and $\frac{360}{1357}$ are examples of rational numbers.

(ii) All natural numbers are rational numbers as $1 = \frac{1}{1}, 2 = \frac{2}{1}, 7 = \frac{7}{1}$, and so on.

But all rational numbers are not natural numbers. For example, $\frac{1}{5}$ and $\frac{-7}{3}$ are rational numbers but are not natural numbers.

(iii) Zero is also a rational number since $0 = \frac{0}{\pm 1} = \frac{0}{\pm 2} = \frac{0}{\pm 3} = \ldots$.

(iv) All whole numbers are rational numbers as $0 = \frac{0}{\pm 1}, 3 = \frac{3}{1}, 8 = \frac{8}{1}$, and so on. But all rational numbers are not whole numbers. For example, $\frac{1}{7}$ and $\frac{-3}{17}$ are rational numbers but are not whole numbers.

(v) All integers are rational numbers because $3 = \frac{3}{1}, -19 = \frac{-19}{1}$, $73 = \frac{73}{1}$, and so on.

But every rational number is not an integer. For example, the rational numbers $\frac{2}{3}$ and $\frac{-6}{7}$ are not integers.

(vi) All terminating as well as recurring decimals are rational numbers.

For example, $1.79 = \frac{179}{100}$ and $0.\dot{2} = \frac{2}{9}$ are rational numbers.

Also, all rational numbers can be expressed as either a terminating decimal or a recurring decimal. Numbers such as $\sqrt{2}$, $\sqrt{3}$ and π cannot be expressed as rational numbers. They are called irrational numbers or simply irrationals.

Equivalent (or equal) rational numbers

If the numerator and the denominator of a rational number are multiplied or divided by the same nonzero number, we get an equivalent rational number. The value of the rational number remains unchanged; so a pair of such rational numbers are often called equal rational numbers.

Symbolically, if $\frac{p}{q}$ is a rational number and $m \neq 0$ then

$$\frac{p}{q} = \frac{p \times m}{q \times m} \text{ and } \frac{p}{q} = \frac{p \div m}{q \div m}.$$

Here, $\frac{p}{q}$ and $\frac{p \times m}{q \times m}$ $\left(\text{or } \frac{p}{q} \text{ and } \frac{p \div m}{q \div m}\right)$ are equivalent, or equal, rational numbers.

Examples (i) $\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}$, $\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$, $\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}$, and so on.

$\therefore \frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{15}{20} = \ldots$

(ii) $\frac{-24}{30} = \frac{-24 \div 2}{30 \div 2} = \frac{-12}{15}$, $\frac{-24}{30} = \frac{-24 \div 3}{30 \div 3} = \frac{-8}{10}$,

$\frac{-24}{30} = \frac{-24 \div 6}{30 \div 6} = \frac{-4}{5}$, etc.

$\therefore \frac{-24}{30} = \frac{-12}{15} = \frac{-8}{10} = \frac{-4}{5} = \ldots$

Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if $ad = bc$.

Examples (i) The rational numbers $\frac{-3}{7}$ and $\frac{21}{-49}$ are equivalent because $(-3) \times (-49) = 7 \times 21$.

(ii) The rational numbers $\frac{7}{15}$ and $\frac{12}{13}$ are not equivalent because $7 \times 13 \neq 15 \times 12$.

Standard form of a rational number

A rational number is said to be in its standard form (or simplest form) if its numerator and denominator have no common factors except 1.

Examples $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{-11}{13}$ are in their standard forms but $\frac{2}{4}$, $\frac{4}{-12}$ and $\frac{-30}{129}$ are not in their standard forms.

A rational number can be reduced to its standard form by cancelling out the common factors in the numerator and the denominator.

Examples (i) $\frac{2}{4} = \frac{2 \times 1}{2 \times 2} = \frac{1}{2}$

(ii) $\frac{4}{-12} = \frac{2 \times 2}{-2 \times 2 \times 3} = \frac{-1}{3}$

Operations on Rational Numbers

Addition of rational numbers

Case I When the given rational numbers have the same denominator, proceed as the following.

Let $\frac{a}{b}$ and $\frac{c}{b}$ be any two given rational numbers. Then, their sum is $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

Similarly, the sum of $\frac{a}{b}$, $\frac{c}{b}$ and $\frac{d}{b}$ is $\frac{a}{b} + \frac{c}{b} + \frac{d}{b} = \frac{a+c+d}{b}$.

Sum of rational numbers with the same denominator $= \frac{\text{sum of the numerators}}{\text{common denominator}}$

Note If the denominator of a rational number is negative, convert it into a rational number with a positive denominator.

EXAMPLE **Find the sum of each of the following.**

(i) $\frac{3}{8} + \frac{5}{8}$ (ii) $\frac{2}{-3} + \frac{7}{3}$ (iii) $1\frac{4}{5} + \frac{1}{-5}$

Solution

(i) $\frac{3}{8} + \frac{5}{8} = \frac{3+5}{8} = \frac{8}{8} = 1$

(ii) $\frac{2}{-3} = \frac{2 \times (-1)}{(-3) \times (-1)} = \frac{-2}{3}$

$\frac{2}{-3} + \frac{7}{3} = \frac{-2}{3} + \frac{7}{3} = \frac{-2+7}{3} = \frac{5}{3}$.

(iii) $1\frac{4}{5} = \frac{9}{5}$ and $\frac{1}{-5} = \frac{1 \times (-1)}{(-5) \times (-1)} = \frac{-1}{5}$.

$1\frac{4}{5} + \frac{1}{-5} = \frac{9}{5} + \frac{-1}{5} = \frac{9 + (-1)}{5} = \frac{8}{5}$

Case II When the denominators of the given rational numbers are different, take the following steps.

Steps 1. Find the LCM of the denominators of the given rational numbers.

2. Find the equivalent rational numbers of the given rational numbers with the LCM as their common denominator.

3. Add the numbers as in Case I.

EXAMPLE **Add $\frac{2}{5}$ and $\frac{1}{3}$.**

Solution The denominators of the given rational numbers are 5 and 3.

The LCM of 5 and 3 is $5 \times 3 = 15$.

Now, $\frac{2}{5} = \frac{2 \times 3}{5 \times 3} = \frac{6}{15}$ and $\frac{1}{3} = \frac{1 \times 5}{3 \times 5} = \frac{5}{15}$.

$\therefore \frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{6+5}{11} = \frac{11}{15}$.

Alternative method

The LCM of 5 and 3 is 15.

$$\therefore \frac{2}{5} + \frac{1}{3} = \frac{2 \times (\text{LCM} \div \text{denominator of the first number}) + 1 \times (\text{LCM} \div \text{denominator of the second number})}{15}$$

$$= \frac{2 \times (15 \div 5) + 1 \times (15 \div 3)}{15}$$

$$= \frac{2 \times 3 + 1 \times 5}{15} = \frac{6+5}{15} = \frac{11}{15}.$$

EXAMPLE **Find $\frac{-7}{12} + \frac{5}{9}$.**

Solution The LCM of 12 and 9 is 36.

$$\therefore \frac{-7}{12} + \frac{5}{9} = \frac{-7 \times (\text{LCM} \div 12) + 5 \times (\text{LCM} \div 9)}{36}$$

$$= \frac{-7 \times (36 \div 12) + 5 \times (36 \div 9)}{36}$$

$$= \frac{-7 \times 3 + 5 \times 4}{36} = \frac{-21 + 20}{36} = \frac{-1}{36}.$$

$$\begin{array}{r|l} 3 & 12, \; 9 \\ \hline & 4, \; 3 \end{array}$$

$\therefore$ LCM $= 3 \times 4 \times 3 = 36$.

Properties of the addition of rational numbers

Closure property Addition is closed; that is, the sum of any two rational numbers is again a rational number. If $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers then $\frac{a}{b} + \frac{c}{d}$ too is a rational number.

Examples (i) $\frac{5}{3}$ and $\frac{6}{3}$ are two rational numbers, and their sum is $\frac{5}{3} + \frac{6}{3} = \frac{5+6}{3} = \frac{11}{3}$, which is also a rational number.

(ii) $\frac{-2}{7}$ and $\frac{1}{3}$ are two rational numbers, and their sum $\frac{-2}{7} + \frac{1}{3} = \frac{-6+7}{21} = \frac{1}{21}$ is also a rational number.

Commutative property Addition is commutative; that is, two rational numbers can be added in any order. In other words, if $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers then $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$

Examples (i) $\frac{1}{4}+\frac{2}{7}=\frac{7+8}{28}=\frac{15}{28}$.

Also, $\frac{2}{7}+\frac{1}{4}=\frac{8+7}{28}=\frac{15}{28}$.

$\therefore \quad \frac{1}{4}+\frac{2}{7}=\frac{2}{7}+\frac{1}{4}$.

(ii) $\frac{2}{3}+\frac{-5}{6}=\frac{4+(-5)}{6}=\frac{-1}{6}$.

Again, $\frac{-5}{6}+\frac{2}{3}=\frac{-5+4}{6}=\frac{-1}{6}$.

$\therefore \quad \frac{2}{3}+\frac{-5}{6}=\frac{-5}{6}+\frac{2}{3}$.

Associative property Addition is associative; that is, if $\frac{a}{b}, \frac{c}{d}$ and $\frac{e}{f}$ are any three rational numbers then $\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}$.

Examples (i) $\frac{1}{3}+\left(\frac{5}{7}+\frac{2}{5}\right)=\frac{1}{3}+\frac{25+14}{35}=\frac{1}{3}+\frac{39}{35}=\frac{35+117}{105}=\frac{152}{105}$.

Again, $\left(\frac{1}{3}+\frac{5}{7}\right)+\frac{2}{5}=\frac{7+15}{21}+\frac{2}{5}=\frac{22}{21}+\frac{2}{5}=\frac{110+42}{105}=\frac{152}{105}$.

$\therefore \quad \frac{1}{3}+\left(\frac{5}{7}+\frac{2}{5}\right)=\left(\frac{1}{3}+\frac{5}{7}\right)+\frac{2}{5}$.

(ii) $-\frac{1}{2}+\left(\frac{2}{3}+\frac{3}{5}\right)=-\frac{1}{2}+\frac{10+9}{15}=-\frac{1}{2}+\frac{19}{15}=\frac{-15+38}{30}=\frac{23}{30}$.

Also, $\left(-\frac{1}{2}+\frac{2}{3}\right)+\frac{3}{5}=\frac{-3+4}{6}+\frac{3}{5}=\frac{1}{6}+\frac{3}{5}=\frac{5+18}{30}=\frac{23}{30}$.

$\therefore \quad -\frac{1}{2}+\left(\frac{2}{3}+\frac{3}{5}\right)=\left(-\frac{1}{2}+\frac{2}{3}\right)+\frac{3}{5}$.

Existence of the additive identity Zero is a rational number such that the sum of any rational number $\frac{a}{b}$ and 0 is the rational number $\frac{a}{b}$. So, $\frac{a}{b}+0=0+\frac{a}{b}=\frac{a}{b}$.

Zero (0) is called the additive identity for rational numbers.

Examples (i) $\frac{2}{3}+0=\frac{2}{3}$ and $0+\frac{2}{3}=\frac{2}{3}$.

$\therefore \frac{2}{3}+0=0+\frac{2}{3}=\frac{2}{3}$.

(ii) $\frac{-7}{11}+0=\frac{-7}{11}$ and $0+\frac{-7}{11}=\frac{-7}{11}$.

$\therefore \frac{-7}{11}+0=0+\frac{-7}{11}=\frac{-7}{11}$.

Existence of the additive inverse For each rational number $\frac{a}{b}$, there exists another rational number $-\frac{a}{b}$ such that their sum is zero; that is, $\frac{a}{b}+\left(-\frac{a}{b}\right)=\left(-\frac{a}{b}\right)+\frac{a}{b}=0$. Here, $-\frac{a}{b}$ is called the additive inverse of $\frac{a}{b}$, and $\frac{a}{b}$ is called the additive inverse of $-\frac{a}{b}$.

Examples (i) $\frac{2}{3}+\frac{-2}{3}=\frac{2+(-2)}{3}=\frac{0}{3}=0$.

Hence, $\frac{-2}{3}$ is called the additive inverse of $\frac{2}{3}$, and $\frac{2}{3}$ is called the additive inverse of $\frac{-2}{3}$.

(ii) $\frac{-18}{79}+\frac{18}{79}=\frac{-18+18}{79}=\frac{0}{79}=0$.

Also, $\frac{18}{79}+\frac{-18}{79}=\frac{18+(-18)}{79}=\frac{0}{79}=0$.

Thus, $\frac{18}{79}$ and $\frac{-18}{79}$ are the additive inverses of each other.

(iii) $0+0=0$.

So, zero (0) is its own additive inverse.

Subtraction of one rational number from another

Consider any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$.

$$\text{Now, } \frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(\text{additive inverse of } \frac{c}{d}\right)$$

$$= \frac{a}{b} + \left(-\frac{c}{d}\right).$$

$$\text{Also, } \frac{a}{b} - \left(-\frac{c}{d}\right) = \frac{a}{b} + \left(\text{additive inverse of } -\frac{c}{d}\right)$$

$$= \frac{a}{b} + \frac{c}{d}.$$

EXAMPLE **Subtract $\frac{5}{6}$ from $\frac{11}{6}$.**

Solution The additive inverse of $\frac{5}{6}$ is $-\frac{5}{6}$.

$$\therefore \frac{11}{6} - \frac{5}{6} = \frac{11}{6} + \left(\text{additive inverse of } \frac{5}{6}\right)$$

$$= \frac{11}{6} + \left(-\frac{5}{6}\right) = \frac{11 + (-5)}{6} = \frac{6}{6} = 1.$$

EXAMPLE **Subtract $-\frac{1}{3}$ from $\frac{1}{2}$.**

Solution The additive inverse of $-\frac{1}{3}$ is $\frac{1}{3}$.

$$\therefore \frac{1}{2} - \left(-\frac{1}{3}\right) = \frac{1}{2} + \left(\text{additive inverse of } -\frac{1}{3}\right)$$

$$= \frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = \frac{5}{6}.$$

Properties of the subtraction of one rational number from another

Closure property Subtraction is closed; that is, the difference of two rational numbers is again a rational number. In other words, if $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers then $\frac{a}{b} - \frac{c}{d}$ is also a rational number.

Examples (i) $\frac{1}{3}$ and $\frac{2}{11}$ are rational numbers. Their difference $\frac{1}{3} - \frac{2}{11} = \frac{1}{3} + \left(\frac{-2}{11}\right)$

$$= \frac{11 + (-6)}{33} = \frac{5}{33} \text{ is also a rational number.}$$

(ii) $\frac{-7}{5}$ and $\frac{-3}{2}$ are rational numbers. Their difference is $\frac{-7}{5}-\left(\frac{-3}{2}\right)=\frac{-7}{5}+\frac{3}{2}=\frac{-14+15}{10}=\frac{1}{10}$, which is also a rational number.

Commutative property Subtraction is not commutative; that is, if $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers then $\frac{a}{b}-\frac{c}{d}\neq\frac{c}{d}-\frac{a}{b}$.

Example $\frac{1}{7}-\frac{3}{11}=\frac{11-21}{77}=\frac{-10}{77}$ and $\frac{3}{11}-\frac{1}{7}=\frac{21-11}{77}=\frac{10}{77}$.

$\therefore \frac{1}{7}-\frac{3}{11}\neq\frac{3}{11}-\frac{1}{7}$.

Associative property Subtraction is not associative; that is, if $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ are any three rational numbers then $\frac{a}{b}-\left(\frac{c}{d}-\frac{e}{f}\right)\neq\left(\frac{a}{b}-\frac{c}{d}\right)-\frac{e}{f}$.

Example $\frac{1}{3}-\left(\frac{3}{5}-\frac{1}{2}\right)=\frac{1}{3}-\frac{6-5}{10}=\frac{1}{3}-\frac{1}{10}=\frac{10-3}{30}=\frac{7}{30}$.

But $\left(\frac{1}{3}-\frac{3}{5}\right)-\frac{1}{2}=\frac{5-9}{15}-\frac{1}{2}=\frac{-4}{15}-\frac{1}{2}=\frac{-8-15}{30}=\frac{-23}{10}$.

So, $\frac{1}{3}-\left(\frac{3}{5}-\frac{1}{2}\right)\neq\left(\frac{1}{3}-\frac{3}{5}\right)-\frac{1}{2}$.

Solved Examples

EXAMPLE 1 **Find the sum of $1\frac{1}{15}$ and $\frac{7}{-10}$.**

Solution $1\frac{1}{15}+\frac{7}{-10}=\frac{16}{15}+\frac{-7}{10}$

$$=\frac{16\times(30\div15)+(-7)\times(30\div10)}{30}$$

$$=\frac{16\times2+(-7)\times3}{30}=\frac{32+(-21)}{30}=\frac{11}{30}.$$

EXAMPLE 2 **Find the value of $\frac{3}{7}+\frac{4}{3}+\frac{-2}{7}+\frac{-11}{3}$.**

Solution Taking the rational numbers with the same denominator together, we have

$$\frac{3}{7}+\frac{4}{3}+\frac{-2}{7}+\frac{-11}{3}=\left(\frac{3}{7}+\frac{-2}{7}\right)+\left(\frac{4}{3}+\frac{-11}{3}\right)$$
$$=\frac{3+(-2)}{7}+\frac{4+(-11)}{3}$$
$$=\frac{1}{7}+\frac{-7}{3}=\frac{1\times3}{7\times3}+\frac{-7\times7}{3\times7}$$
$$=\frac{3}{21}+\frac{-49}{21}=\frac{3+(-49)}{21}=\frac{-46}{21}.$$

EXAMPLE 3 **Simplify $\frac{3}{5}+\frac{7}{6}+\frac{-2}{3}+\frac{-7}{10}$.**

Solution

$$\frac{3}{5}+\frac{7}{6}+\frac{-2}{3}+\frac{-7}{10}=\left(\frac{3}{5}+\frac{-7}{10}\right)+\left(\frac{7}{6}+\frac{-2}{3}\right)$$
$$=\frac{6+(-7)}{10}+\frac{7+(-4)}{6}=\frac{-1}{10}+\frac{3}{6}$$
$$=\frac{-1}{10}+\frac{1}{2}=\frac{-1+5}{10}=\frac{4}{10}=\frac{2}{5}.$$

EXAMPLE 4 **The sum of two rational numbers is $\frac{-2}{3}$. If one of the numbers is $\frac{-5}{6}$, find the other.**

Solution Let the other number be x.

$$\therefore \quad \frac{-5}{6}+x=\frac{-2}{3}$$
$$\Rightarrow \quad x=\frac{-2}{3}+\left(\text{additive inverse of } \frac{-5}{6}\right)$$
$$\Rightarrow \quad x=\frac{-2}{3}+\frac{5}{6}=\frac{-4+5}{6}=\frac{1}{6}.$$

Hence, the other rational number is $\frac{1}{6}$.

EXAMPLE 5 **What number should be added to $\frac{-3}{4}$ to get $\frac{-5}{14}$?**

Solution Let the required number be x.

$$\therefore \quad x+\frac{-3}{4}=\frac{-5}{14}$$
$$\Rightarrow \quad x=\frac{-5}{14}+\left(\text{additive inverse of } \frac{-3}{4}\right)$$
$$=\frac{-5}{14}+\frac{3}{4}$$
$$=\frac{-10+21}{28}=\frac{11}{28}.$$

Hence, the required number is $\frac{11}{28}$.

EXAMPLE 6 **What number should be subtracted from $-\frac{3}{2}$ to get $-\frac{1}{3}$?**

Solution Let the required number be x.

$\therefore \quad -\frac{3}{2} - x = -\frac{1}{3}$

$\Rightarrow \quad x = \frac{-3}{2} + \left(\text{additive inverse of } -\frac{1}{3}\right)$

$= \frac{-3}{2} + \frac{1}{3} = \frac{-9+2}{6} = \frac{-7}{6}.$

Hence, the required number is $\frac{-7}{6}$.

EXERCISE 1A

1. Add each of the following pairs of rational numbers.

(i) $-\frac{1}{3}$ and $\frac{2}{3}$ (ii) $\frac{2}{5}$ and $\frac{-3}{5}$ (iii) $\frac{-7}{11}$ and $\frac{-4}{11}$ (iv) $\frac{-13}{17}$ and $\frac{4}{17}$

(v) $\frac{11}{25}$ and $\frac{-7}{25}$ (vi) $\frac{-8}{9}$ and $\frac{-19}{9}$

2. Add.

(i) $\frac{2}{3}$ and $\frac{3}{5}$ (ii) $\frac{-2}{5}$ and $\frac{5}{7}$ (iii) $\frac{3}{-8}$ and $\frac{-5}{12}$ (iv) $\frac{-7}{26}$ and $\frac{-5}{39}$

(v) $\frac{5}{-24}$ and $\frac{7}{36}$ (vi) $\frac{3}{16}$ and $\frac{-7}{24}$ (vii) $4\frac{11}{25}$ and $\frac{13}{-15}$ (viii) -2 and $1\frac{7}{11}$

(ix) -1 and $2\frac{3}{5}$ (x) $\frac{-7}{8}$ and 0

3. Verify the commutative law of addition for the following pairs of rational numbers.

(i) $\frac{-5}{7}$ and $\frac{3}{4}$ (ii) $\frac{-1}{3}$ and $\frac{-2}{5}$ (iii) $\frac{4}{-7}$ and $\frac{-2}{21}$ (iv) 5 and $\frac{-3}{5}$

4. Verify the associative law of addition for the following groups of rational numbers.

(i) $\frac{1}{2}, \frac{-2}{3}$ and $\frac{1}{5}$ (ii) $\frac{-3}{5}, \frac{7}{10}$ and $\frac{4}{15}$ (iii) $\frac{3}{-7}, \frac{2}{21}$ and $\frac{-5}{14}$ (iv) $1, \frac{5}{-11}$ and $\frac{7}{22}$

5. Find the additive inverse of each of the following rationals.

(i) 0 (ii) 1 (iii) -5 (iv) $\frac{-3}{4}$

(v) $\frac{2}{-9}$ (vi) $\frac{-7}{-15}$ (vii) $\frac{-13}{2}$ (viii) $\frac{-25}{-133}$

(ix) $\frac{13}{-27}$ (x) $\frac{17}{8}$

6. Using the appropriate properties of addition, find the value of each of the following sums.

(i) $\frac{5}{7} + \frac{-7}{3} + \frac{-3}{7} + \frac{11}{3}$ (ii) $\frac{4}{5} + \frac{3}{10} + \frac{11}{5} + \frac{-13}{10}$

(iii) $\frac{2}{3}+\frac{-11}{8}+\frac{-17}{3}+\frac{3}{8}$ (iv) $\frac{7}{8}+\frac{3}{10}+\frac{-5}{8}+\frac{-7}{20}$

7. Fill in the blanks.

(i) $\frac{-2}{15}+\frac{-3}{19}=\frac{-3}{19}+\ldots\ldots$ (ii) $\frac{23}{12}+\frac{7}{5}=\frac{7}{5}+\ldots\ldots$

(iii) $\frac{1}{2}+\left(\frac{3}{5}+\frac{-7}{9}\right)=\left(\frac{1}{2}+\ldots\ldots\right)+\frac{-7}{9}$ (iv) $\left(\frac{-5}{11}+\frac{-7}{13}\right)+\frac{25}{29}=\ldots\ldots+\left(\frac{-7}{13}+\frac{25}{29}\right)$

(v) $-3+\left(\frac{-5}{23}+\frac{7}{31}\right)=\left(\ldots\ldots+\frac{-5}{23}\right)+\frac{7}{31}$

8. Subtract.

(i) $\frac{1}{3}$ from $\frac{2}{5}$ (ii) $\frac{-7}{8}$ from $\frac{1}{2}$ (iii) $\frac{-11}{13}$ from $\frac{-2}{3}$ (iv) $\frac{-13}{11}$ from -1

(v) $\frac{-15}{17}$ from 1 (vi) $\frac{12}{19}$ from -2 (vii) $\frac{-19}{31}$ from 0 (viii) -5 from $\frac{-2}{3}$

(ix) 7 from $\frac{-7}{17}$ (x) $\frac{-9}{8}$ from $\frac{-13}{7}$

9. Verify.

(i) $\frac{1}{3}-\frac{1}{4}\neq\frac{1}{4}-\frac{1}{3}$ (ii) $\frac{-2}{7}-\frac{3}{5}\neq\frac{3}{5}-\frac{-2}{7}$

(iii) $\frac{1}{2}-\left(\frac{1}{3}-\frac{1}{5}\right)\neq\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{1}{5}$ (iv) $\frac{5}{11}-\left(\frac{-2}{3}-\frac{5}{7}\right)\neq\left\{\frac{5}{11}-\left(\frac{-2}{3}\right)\right\}-\frac{5}{7}$

10. (a) The sum of two rational numbers is $\frac{-3}{4}$. If one of the numbers is $-\frac{28}{3}$, find the other.

(b) The sum of two rational numbers is -3. If one of the numbers is $\frac{7}{8}$, find the other.

11. (a) What number should be added to $\frac{-11}{4}$ to get $\frac{-25}{14}$?

(b) What number should be added to -2 to get $\frac{-5}{8}$?

12. (a) What number should be subtracted from $\frac{-7}{8}$ to get $\frac{-5}{12}$?

(b) What number should be subtracted from 1 to get $-\frac{7}{17}$?

13. Identify the true statements only.

(i) The sum of two rational numbers is a rational number.

(ii) The difference of two rational numbers is not always a rational number.

(iii) Zero is not a rational number.

(iv) Addition is commutative on rational numbers.

(v) Subtraction is commutative on rational numbers.

(vi) Addition is associative on rational numbers.

(vii) Subtraction is associative on rational numbers.

(viii) Zero is the additive inverse of itself.

(ix) Zero is the additive identity for rational numbers.

(x) 1 is the additive inverse of -1.

ANSWERS

1. (i) $\frac{1}{3}$ (ii) $\frac{-1}{5}$ (iii) -1 (iv) $\frac{-9}{17}$ (v) $\frac{4}{25}$ (vi) -3

2. (i) $\frac{19}{15}$ (ii) $\frac{11}{35}$ (iii) $\frac{-19}{24}$ (iv) $\frac{-31}{78}$ (v) $\frac{-1}{72}$ (vi) $\frac{-5}{48}$ (vii) $\frac{268}{75}$ (viii) $-\frac{4}{11}$ (ix) $\frac{8}{5}$ (x) $\frac{-7}{8}$

5. (i) 0 (ii) -1 (iii) 5 (iv) $\frac{3}{4}$ (v) $\frac{2}{9}$ (vi) $\frac{-7}{15}$ (vii) $\frac{13}{2}$ (viii) $\frac{-25}{133}$ (ix) $\frac{13}{27}$ (x) $\frac{-17}{8}$

6. (i) $\frac{34}{21}$ (ii) 2 (iii) -6 (iv) $\frac{1}{5}$

7. (i) $\frac{-2}{15}$ (ii) $\frac{23}{12}$ (iii) $\frac{3}{5}$ (iv) $\frac{-5}{11}$ (v) -3

8. (i) $\frac{1}{15}$ (ii) $\frac{11}{8}$ (iii) $\frac{7}{39}$ (iv) $\frac{2}{11}$ (v) $\frac{32}{17}$ (vi) $\frac{-50}{19}$ (vii) $\frac{19}{31}$ (viii) $\frac{13}{3}$ (ix) $\frac{-126}{17}$ (x) $\frac{-41}{56}$

10. (a) $\frac{103}{12}$ (b) $\frac{-31}{8}$

11. (a) $\frac{27}{28}$ (b) $\frac{11}{8}$

12. (a) $\frac{-11}{24}$ (b) $\frac{24}{17}$

13. (i), (iv), (vi), (viii), (ix) and (x)

Multiplication of rational numbers

$$\text{Product of two (or more) rational numbers} = \frac{\text{product of their numerators}}{\text{product of their denominators}}.$$

If $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers then

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

EXAMPLE **Find the value of each of the following products.**

(i) $\frac{1}{3} \times \frac{2}{7}$ **(ii)** $\frac{-12}{35} \times \frac{14}{15}$ **(iii)** $\frac{-3}{5} \times \frac{-10}{9}$

Solution (i) $\frac{1}{3} \times \frac{2}{7} = \frac{1 \times 2}{3 \times 7} = \frac{2}{21}$.

(ii) $\frac{-12}{35} \times \frac{14}{15} = \frac{(-12) \times 14}{35 \times 15} = \frac{-(\cancel{12}^{4} \times \cancel{14}^{2})}{\cancel{35}_{5} \times \cancel{15}_{5}} = \frac{-8}{25}$.

(iii) $\frac{-3}{5} \times \frac{-10}{9} = \frac{(-3) \times (-10)}{5 \times 9} = \frac{\cancel{3}^{1} \times \cancel{10}^{2}}{\cancel{5}_{1} \times \cancel{9}_{3}} = \frac{2}{3}$.

Properties of the multiplication of rational numbers

Closure property Multiplication of rational numbers is closed; that is, the product of two rational numbers is always a rational number. In other words, if $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers then $\frac{a}{b} \times \frac{c}{d}$ too is a rational number.

Examples (i) $\frac{4}{3}$ and $\frac{2}{5}$ are two rational numbers, and their product is $\frac{4}{3}\times\frac{2}{5}=\frac{4\times2}{3\times5}=\frac{8}{15}$, which is also a rational number.

(ii) $\frac{-2}{7}$ and $\frac{-21}{10}$ are rational numbers, and their product $\frac{-2}{7}\times\frac{-21}{10}=\frac{(-2)\times(-21)}{7\times10}=\frac{\cancel{2}^1\times\cancel{21}^3}{\cancel{7}_1\times\cancel{10}_5}=\frac{3}{5}$ is also a rational number.

Commutative property Multiplication is commutative; that is, two rational numbers can be multiplied in any order, and the product remains the same. In other words, if $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers then $\frac{a}{b}\times\frac{c}{d}=\frac{c}{d}\times\frac{a}{b}$.

EXAMPLE **Verify whether $\frac{1}{3}$ and $\frac{5}{7}$ satisfy the commutative law of multiplication.**

Solution $\frac{1}{3}\times\frac{5}{7}=\frac{1\times5}{3\times7}=\frac{5}{21}$ and $\frac{5}{7}\times\frac{1}{3}=\frac{5\times1}{7\times3}=\frac{5}{21}$.

$\therefore\ \frac{1}{3}\times\frac{5}{7}=\frac{5}{7}\times\frac{1}{3}$.

So, the commutative law of multiplication is verified.

EXAMPLE **Verify.**

(i) $\frac{-3}{4}\times\frac{7}{11}=\frac{7}{11}\times\frac{-3}{4}$. (ii) $\frac{-9}{13}\times\frac{-2}{5}=\frac{-2}{5}\times\frac{-9}{13}$.

Solution (i) $\frac{-3}{4}\times\frac{7}{11}=\frac{(-3)\times7}{4\times11}=\frac{-21}{44}$ and $\frac{7}{11}\times\frac{-3}{4}=\frac{7\times(-3)}{11\times4}=\frac{-21}{44}$.

$\therefore\ \frac{-3}{4}\times\frac{7}{11}=\frac{7}{11}\times\frac{-3}{4}$.

(ii) $\frac{-9}{13}\times\frac{-2}{5}=\frac{(-9)\times(-2)}{13\times5}=\frac{18}{65}$ and $\frac{-2}{5}\times\frac{-9}{13}=\frac{(-2)\times(-9)}{5\times13}=\frac{18}{65}$.

$\therefore\ \frac{-9}{13}\times\frac{-2}{5}=\frac{-2}{5}\times\frac{-9}{13}$.

Associative property If $\frac{a}{b},\frac{c}{d}$ and $\frac{e}{f}$ are any three rational numbers then

$$\frac{a}{b}\times\left(\frac{c}{d}\times\frac{e}{f}\right)=\left(\frac{a}{b}\times\frac{c}{d}\right)\times\frac{e}{f}.$$

EXAMPLE **Verify the following.**

(i) $\frac{1}{3}\times\left(\frac{3}{7}\times\frac{2}{5}\right)=\left(\frac{1}{3}\times\frac{3}{7}\right)\times\frac{2}{5}$.

(ii) $\frac{-5}{6}\times\left(\frac{3}{4}\times\frac{1}{2}\right)=\left(\frac{-5}{6}\times\frac{3}{4}\right)\times\frac{1}{2}$.

(iii) $\frac{-2}{5}\times\left[\left(-\frac{3}{10}\right)\times\left(-\frac{4}{15}\right)\right]=\left[\left(-\frac{2}{5}\right)\times\left(-\frac{3}{10}\right)\right]\times\frac{-4}{15}$.

Solution (i) $\frac{1}{3}\times\left(\frac{3}{7}\times\frac{2}{5}\right)=\frac{1}{3}\times\frac{3\times2}{7\times5}=\frac{1}{3}\times\frac{6}{35}=\frac{1\times6}{3\times35}=\frac{2}{35}$

and $\left(\frac{1}{3}\times\frac{3}{7}\right)\times\frac{2}{5}=\frac{1\times3}{3\times7}\times\frac{2}{5}=\frac{1}{7}\times\frac{2}{5}=\frac{1\times2}{7\times5}=\frac{2}{35}$.

$\therefore\ \frac{1}{3}\times\left(\frac{3}{7}\times\frac{2}{5}\right)=\left(\frac{1}{3}\times\frac{3}{7}\right)\times\frac{2}{5}$.

(ii) $\frac{-5}{6}\times\left(\frac{3}{4}\times\frac{1}{2}\right)=\frac{-5}{6}\times\frac{3\times1}{4\times2}=\frac{-5}{6}\times\frac{3}{8}=\frac{-5\times3}{6\times8}=\frac{-5}{16}$

and $\left(\frac{-5}{6}\times\frac{3}{4}\right)\times\frac{1}{2}=\frac{-5\times3}{6\times4}\times\frac{1}{2}=\frac{-5}{8}\times\frac{1}{2}=\frac{-5\times1}{8\times2}=\frac{-5}{16}$.

$\therefore\ \frac{-5}{6}\times\left(\frac{3}{4}\times\frac{1}{2}\right)=\left(\frac{-5}{6}\times\frac{3}{4}\right)\times\frac{1}{2}$.

(iii) $\frac{-2}{5}\times\left\{\left(-\frac{3}{10}\right)\times\left(-\frac{4}{15}\right)\right\}=\frac{-2}{5}\times\frac{(-3)\times(-4)}{10\times15}=\frac{-2}{5}\times\frac{2}{25}=\frac{-2\times2}{5\times25}=\frac{-4}{125}$

and $\left\{\left(-\frac{2}{5}\right)\times\left(-\frac{3}{10}\right)\right\}\times\frac{-4}{15}=\frac{(-2)\times(-3)}{5\times10}\times\frac{-4}{15}=\frac{3}{25}\times\frac{-4}{15}=\frac{3\times(-4)}{25\times15}$

$=\frac{-(3\times4)}{25\times15}=\frac{-4}{125}$.

So, $\frac{-2}{5}\times\left\{\left(-\frac{3}{10}\right)\times\left(-\frac{4}{15}\right)\right\}=\left\{\left(-\frac{2}{5}\right)\times\left(-\frac{3}{10}\right)\right\}\times\frac{-4}{15}$.

Existence of the multiplicative identity 1 is a rational number such that the product of any rational number $\frac{a}{b}$ and 1 is the rational number itself. So, $\frac{a}{b}\times1=1\times\frac{a}{b}=\frac{a}{b}$. Hence, 1 is called the multiplicative identity for rational numbers.

Examples (i) $\frac{2}{5}\times1=\frac{2}{5}\times\frac{1}{1}=\frac{2\times1}{5\times1}=\frac{2}{5}$ and $1\times\frac{2}{5}=\frac{1}{1}\times\frac{2}{5}=\frac{1\times2}{1\times5}=\frac{2}{5}$.

Hence, $\frac{2}{5}\times1=1\times\frac{2}{5}=\frac{2}{5}$.

(ii) $\frac{-7}{11}\times1=\frac{-7}{11}\times\frac{1}{1}=\frac{-7\times1}{11\times1}=\frac{-7}{11}$ and $1\times\frac{-7}{11}=\frac{1}{1}\times\frac{-7}{11}=\frac{1\times(-7)}{1\times11}=\frac{-7}{11}$.

So, $\frac{-7}{11}\times1=1\times\frac{-7}{11}=\frac{-7}{11}$.

Existence of the multiplicative inverse For each nonzero rational number $\frac{a}{b}$, there exists another rational number $\frac{b}{a}$ such that $\frac{a}{b}\times\frac{b}{a}=\frac{b}{a}\times\frac{a}{b}=1$. Hence, $\frac{b}{a}$ is called the multiplicative inverse (also called the reciprocal) of $\frac{a}{b}$. It is mathematically expressed as $\frac{b}{a}=\left(\frac{a}{b}\right)^{-1}$. Conversely, $\frac{a}{b}$ is the multiplicative inverse of $\frac{b}{a}$. In other words, $\frac{a}{b}$ and $\frac{b}{a}$ are the reciprocals of each other.

Examples (i) $\frac{2}{3}\times\frac{3}{2}=\frac{3}{2}\times\frac{2}{3}=1$.

So, $\frac{2}{3}$ is the multiplicative inverse of $\frac{3}{2}$, and $\frac{3}{2}$ is the multiplicative inverse of $\frac{2}{3}$.

(ii) $\frac{-7}{9}\times\frac{-9}{7}=\frac{-9}{7}\times\frac{-7}{9}=1$.

So, $\frac{-7}{9}$ and $\frac{-9}{7}$ are the reciprocals of $\frac{-9}{7}$ and $\frac{-7}{9}$ respectively.

(iii) $1\times1=1$.

So, 1 is the reciprocal of itself.

(iv) $(-1)\times(-1)=1$.

So, -1 is the reciprocal of -1.

(v) Zero (0) has no reciprocal.

Distributive property If $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be any three rational numbers then $\frac{a}{b}\times\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}\times\frac{c}{d}+\frac{a}{b}\times\frac{e}{f}$ and $\left(\frac{c}{d}+\frac{e}{f}\right)\times\frac{a}{b}=\frac{c}{d}\times\frac{a}{b}+\frac{e}{f}\times\frac{a}{b}$.

Example Consider the rational numbers $\frac{1}{2}$, $\frac{-3}{4}$ and $\frac{-5}{7}$.

Now, $\frac{1}{2}\times\left\{\left(-\frac{3}{4}\right)+\left(-\frac{5}{7}\right)\right\}=\frac{1}{2}\times\frac{(-3)\times7+(-5)\times4}{28}$

$=\frac{1}{2}\times\left(-\frac{41}{28}\right)=-\frac{41}{56}$.

Also, $\frac{1}{2}\times\left(-\frac{3}{4}\right)+\frac{1}{2}\times\left(-\frac{5}{7}\right)=\frac{1\times(-3)}{2\times4}+\frac{1\times(-5)}{2\times7}$

$=\frac{-3}{8}+\frac{-5}{14}=\frac{(-3)\times7+(-5)\times4}{56}$

$=\frac{-21+(-20)}{56}=\frac{-41}{56}$.

Thus, $\frac{1}{2}\times\left\{\left(-\frac{3}{4}\right)+\left(-\frac{5}{7}\right)\right\}=\frac{1}{2}\times\left(-\frac{3}{4}\right)+\frac{1}{2}\times\left(-\frac{5}{7}\right)$.

Multiplicative property of zero The product of any rational number $\frac{a}{b}$ and 0 is 0; that is,

$$\frac{a}{b}\times 0=0\times\frac{a}{b}=0.$$

Examples (i) $\frac{2}{11}\times 0=\frac{2}{11}\times\frac{0}{1}=\frac{2\times 0}{11\times 1}=\frac{0}{11}=0$

and $0\times\frac{2}{11}=\frac{0}{1}\times\frac{2}{11}=\frac{0\times 2}{1\times 11}=\frac{0}{11}=0.$

(ii) $\frac{-5}{13}\times 0=\frac{-5}{13}\times\frac{0}{1}=\frac{-5\times 0}{13\times 1}=\frac{0}{13}=0$

and $0\times\frac{-5}{13}=\frac{0}{1}\times\frac{-5}{13}=\frac{0\times(-5)}{1\times 13}=\frac{0}{13}=0.$

Solved Examples

EXAMPLE 1 **Find the reciprocal of each of the following.**

(i) $1\frac{3}{4}$ (ii) $\frac{-17}{8}$ (iii) $\frac{-4}{5}\times\frac{15}{16}$ (iv) $\frac{-14}{9}\times\frac{3}{7}$

Solution (i) $1\frac{3}{4}=\frac{7}{4}$.

Now, the reciprocal of $\frac{7}{4}$ is $\frac{4}{7}$.

(ii) The reciprocal of $\frac{-17}{8}$ is $\frac{-8}{17}$.

(iii) $\frac{-4}{5}\times\frac{15}{16}=\frac{-3}{4}$.

$\therefore$ the reciprocal of $\frac{-4}{5}\times\frac{15}{16}$ is $\frac{-4}{3}$.

(iv) $\frac{-14}{9}\times\frac{3}{7}=\frac{-2}{3}$.

So, the reciprocal of $\frac{-14}{9}\times\frac{3}{7}$ is $\frac{-3}{2}$.

EXAMPLE 2 **Find the value of $\frac{7}{8}\times\frac{3}{7}+\frac{7}{8}\times\left(-\frac{4}{21}\right)+\left(-\frac{5}{24}\right)$ by using the distributive property.**

Solution $\frac{7}{8}\times\frac{3}{7}+\frac{7}{8}\times\left(-\frac{4}{21}\right)+\left(-\frac{5}{24}\right)$

$=\left\{\frac{7}{8}\times\frac{3}{7}+\frac{7}{8}\times\left(-\frac{4}{21}\right)\right\}+\left(-\frac{5}{24}\right)$

$=\frac{7}{8}\times\left\{\frac{3}{7}+\left(-\frac{4}{21}\right)\right\}+\left(-\frac{5}{24}\right)$

$=\frac{7}{8}\times\left\{\frac{9+(-4)}{21}\right\}+\left(-\frac{5}{24}\right)$

$=\frac{\overset{1}{\cancel{7}}}{8}\times\frac{5}{\underset{3}{\cancel{21}}}+\left(-\frac{5}{24}\right)=\frac{5}{24}+\left(-\frac{5}{24}\right)=0.$

EXERCISE 1B

1. Find the value of each of the following products.

(i) $\frac{3}{7}\times\frac{4}{5}$ (ii) $\frac{-4}{9}\times\frac{10}{7}$ (iii) $\frac{-1}{2}\times\frac{-9}{8}$ (iv) $\frac{-3}{4}\times\frac{8}{-15}$

(v) $\frac{21}{-2}\times\frac{-4}{7}$ (vi) $\frac{-7}{10}\times\frac{6}{-35}$ (vii) $\frac{-14}{15}\times\frac{-25}{42}$ (viii) $\frac{-35}{-18}\times\frac{8}{-15}$

(ix) $\frac{-7}{11}\times22$ (x) $\frac{14}{25}\times(-100)$ (xi) $\frac{-13}{-12}\times(-3)$ (xii) $\frac{-12}{85}\times(-17)$

2. Verify the commutative law of multiplication for the following pairs of rational numbers.

(i) $\frac{2}{3}$ and $\frac{-7}{5}$ (ii) $\frac{-1}{7}$ and $\frac{14}{3}$ (iii) $\frac{3}{-8}$ and $\frac{-16}{15}$ (iv) $\frac{11}{13}$ and $\frac{-26}{-55}$

3. Verify the associative law of multiplication for the following groups of rational numbers.

(i) $\frac{2}{3},\frac{-1}{5}$ and $\frac{7}{12}$ (ii) $\frac{-5}{7},\frac{1}{2}$ and $\frac{-21}{20}$

(iii) $\frac{-8}{9},\frac{-3}{4}$ and $\frac{15}{16}$ (iv) $\frac{-3}{11},\frac{-7}{4}$ and $\frac{-1}{2}$

4. Find the multiplicative inverse of each of the following.

(i) 1 (ii) −1 (iii) $\frac{3}{4}$ (iv) $\frac{5}{-7}$

(v) $\frac{-11}{-13}$ (vi) $\frac{-7}{17}$ (vii) $3\frac{1}{4}$ (viii) $\frac{-26}{5}$

(ix) $\frac{3}{4}\times\frac{-8}{9}$ (x) $\frac{5}{13}\times\frac{-26}{-35}$ (xi) $\frac{-13}{-14}\times\frac{70}{-39}$ (xii) $3\frac{1}{3}\times1\frac{1}{2}$

5. Verify each of the following.

(i) $\frac{3}{5}\times\left(\frac{7}{12}+\frac{1}{2}\right)=\frac{3}{5}\times\frac{7}{12}+\frac{3}{5}\times\frac{1}{2}$.

(ii) $\frac{-2}{3}\times\left(\frac{-5}{4}+\frac{4}{7}\right)=\frac{-2}{3}\times\frac{-5}{4}+\frac{-2}{3}\times\frac{4}{7}$.

(iii) $\frac{8}{11}\times\left(\frac{-3}{2}+\frac{-1}{5}\right)=\frac{8}{11}\times\frac{-3}{2}+\frac{8}{11}\times\frac{-1}{5}$.

(iv) $\frac{-3}{2}\times\left(\frac{-5}{7}+\frac{-5}{2}\right)=\frac{-3}{2}\times\frac{-5}{7}+\frac{-3}{2}\times\frac{-5}{2}$.

(v) $\left(\frac{7}{11}+\frac{3}{5}\right)\times\frac{2}{3}=\frac{7}{11}\times\frac{2}{3}+\frac{3}{5}\times\frac{2}{3}$.

6. Using the distributive property, find the value of each of the following.

(i) $\frac{3}{5}\times\frac{-20}{9}+\frac{3}{5}\times\frac{-1}{3}$ (ii) $\frac{7}{15}\times\frac{-50}{49}+\frac{7}{15}\times\frac{1}{49}$

(iii) $\frac{-13}{5}\times\frac{16}{7}+\frac{-13}{5}\times\frac{19}{7}$ (iv) $\frac{-1}{3}\times\frac{-3}{5}+\frac{-1}{3}\times\frac{7}{11}$

(v) $\frac{2}{3}\times\frac{9}{10}+\frac{2}{3}\times\frac{-4}{10}-\frac{1}{3}$

7. Fill in the blanks.

(i) $\frac{-7}{17}\times\frac{27}{35}=\frac{27}{35}\times$ ……

(ii) $\frac{3}{29} \times \left(\frac{-7}{31} \times \frac{-9}{35}\right) = \left(\frac{3}{29} \times \ldots\ldots\right) \times \frac{-9}{35}$.

(iii) $\frac{-11}{13} \times \left(\frac{17}{47} \times \frac{-3}{5}\right) = \left(\frac{-11}{13} \times \frac{17}{47}\right) \times \ldots\ldots$.

(iv) $\frac{3}{5} \times \left(\frac{-7}{27} + \frac{9}{49}\right) = \frac{3}{5} \times \frac{-7}{27} + \ldots\ldots$.

(v) $\frac{13}{15} \times \left(\ldots\ldots + \frac{-5}{6}\right) = \frac{13}{15} \times \frac{-3}{2} + \frac{13}{15} \times \frac{-5}{6}$.

(vi) $\ldots\ldots \times \left(\frac{7}{3} + \frac{-5}{11}\right) = \frac{2}{3} \times \frac{7}{3} + \frac{2}{3} \times \frac{-5}{11}$.

(vii) The reciprocal of $\frac{2}{3}$ is ....

(viii) $\left(\frac{7}{8}\right)^{-1} = \ldots\ldots$.

(ix) The reciprocal of -1 is ....

(x) -2 is the reciprocal of ....

(xi) The product of a nonzero rational number and its reciprocal is ....

(xii) The reciprocal of a positive rational number is ....

(xiii) The reciprocal of a negative rational number is ....

(xiv) ...... is the only rational number whose reciprocal is not defined.

8. Identify the false statements only.

(i) The product of two rational numbers is not necessarily a rational number.

(ii) Multiplication of rational numbers obeys the commutative property.

(iii) Multiplication of rational numbers obeys the associative property.

(iv) Zero is the multiplicative identity for rational numbers.

(v) The product of a rational number and its multiplicative inverse is 1.

(vi) If $\frac{a}{b}$ is a rational number then $\frac{b}{a} = \left(\frac{a}{b}\right)^{-1}$.

(vii) The reciprocal of $\frac{5}{-6}$ is $\frac{-5}{6}$.

(viii) Zero has no reciprocal.

ANSWERS

1. (i) $\frac{12}{35}$ (ii) $\frac{-40}{63}$ (iii) -6 (iv) $\frac{2}{5}$ (v) 6 (vi) $\frac{3}{25}$ (vii) $\frac{5}{9}$ (viii) $\frac{-28}{27}$ (ix) -14 (x) -56 (xi) $\frac{-13}{4}$ (xii) $\frac{12}{5}$

4. (i) 1 (ii) -1 (iii) $\frac{4}{3}$ (iv) $\frac{-7}{5}$ (v) $\frac{13}{11}$ (vi) $\frac{-17}{7}$ (vii) $\frac{4}{13}$ (viii) $\frac{-5}{26}$ (ix) $\frac{-3}{2}$ (x) $\frac{7}{2}$ (xi) $\frac{-3}{5}$ (xii) $\frac{1}{5}$

6. (i) $\frac{-23}{15}$ (ii) $\frac{-7}{15}$ (iii) -13 (iv) $\frac{-2}{165}$ (v) 0

7. (i) $\frac{-7}{17}$ (ii) $\frac{-7}{31}$ (iii) $\frac{-3}{5}$ (iv) $\frac{3}{5} \times \frac{9}{49}$ (v) $\frac{-3}{2}$ (vi) $\frac{2}{3}$ (vii) $\frac{3}{2}$ (viii) $\frac{8}{7}$ (ix) -1 (x) $-\frac{1}{2}$ (xi) 1 (xii) a positive rational number (xiii) a negative rational number (xiv) 0

8. (i), (iv) and (vii)

Division of one rational number by another

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers such that $\frac{c}{d} \neq 0$. Then, we have

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \left(\text{reciprocal of } \frac{c}{d}\right) = \frac{a}{b} \times \frac{d}{c}.$$

Examples (i) $\frac{14}{15} \div \frac{8}{9} = \frac{14}{15} \times \left(\text{reciprocal of } \frac{8}{9}\right)$

$$= \frac{14}{15} \times \frac{9}{8} = \frac{\cancel{14}^{7} \times \cancel{9}^{3}}{\cancel{15}_{5} \times \cancel{8}_{4}} = \frac{21}{20}.$$

(ii) $\frac{-12}{7} \div \frac{15}{8} = \frac{-12}{7} \times \frac{8}{15} = \frac{-\cancel{12}^{4} \times 8}{7 \times \cancel{15}_{5}} = \frac{-32}{35}.$

(iii) $\frac{2}{3} \div \left(-\frac{7}{9}\right) = \frac{2}{3} \times \left(-\frac{9}{7}\right) = \frac{-2 \times \cancel{9}^{3}}{\cancel{3}_{1} \times 7} = \frac{-6}{7}.$

(iv) $\frac{-27}{16} \div \left(-\frac{9}{10}\right) = \frac{-27}{16} \times \left(-\frac{10}{9}\right) = \frac{\cancel{27}^{3} \times \cancel{10}^{5}}{\cancel{16}_{8} \times \cancel{9}_{1}} = \frac{15}{8}.$

Notes
- $\frac{a}{b} \div 1 = \frac{a}{b}.$
- $\frac{a}{b} \div 0$ is not defined.

Properties of the division of one rational number by another

Closure property Division is closed; that is, if $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers such that $\frac{c}{d} \neq 0$ then $\frac{a}{b} \div \frac{c}{d}$ is also a rational number.

Examples (i) $\frac{2}{3}$ and $\frac{5}{6}$ are two rationals, and $\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \times \frac{6}{5} = \frac{4}{5}$ is also a rational number.

(ii) $-\frac{1}{2}$ and $\frac{3}{4}$ are two rationals, and $-\frac{1}{2} \div \frac{3}{4} = \frac{-1}{2} \times \frac{4}{3} = \frac{-2}{3}$ is also a rational number.

(iii) -2 and $-\frac{4}{7}$ are two rationals, and $-2 \div \frac{-4}{7} = -2 \times \frac{-7}{4} = \frac{7}{2}$ is also a rational number.

Commutative property Division is not commutative; that is, if $\frac{a}{b}$ and $\frac{c}{d}$ be two nonzero rational numbers then $\frac{a}{b} \div \frac{c}{d} \neq \frac{c}{d} \div \frac{a}{b}.$

Examples (i) $2 \div \frac{3}{5} = 2 \times \frac{5}{3} = \frac{10}{3}$ and $\frac{3}{5} \div 2 = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$.

$\therefore \quad 2 \div \frac{3}{5} \neq \frac{3}{5} \div 2.$

(ii) $\frac{14}{9} \div \left(-\frac{7}{3}\right) = \frac{14^2}{9_3} \times \frac{-3^1}{7_1} = -\frac{2}{3}$ and $\left(-\frac{7}{3}\right) \div \frac{14}{9} = \frac{-7^1}{3_1} \times \frac{9^3}{14_2} = -\frac{3}{2}$.

$\therefore \quad \frac{14}{9} \div \left(-\frac{7}{3}\right) \neq \left(-\frac{7}{3}\right) \div \frac{14}{9}.$

Associative property Division is not associative; that is, if $\frac{a}{b}, \frac{c}{d}$ and $\frac{e}{f}$ are three rational numbers such that $\frac{c}{d} \neq 0$ and $\frac{e}{f} \neq 0$ then $\frac{a}{b} \div \left(\frac{c}{d} \div \frac{e}{f}\right) \neq \left(\frac{a}{b} \div \frac{c}{d}\right) \div \frac{e}{f}$.

Example $\frac{1}{2} \div \left(\frac{3}{4} \div \frac{5}{6}\right) = \frac{1}{2} \div \left(\frac{3}{4_2} \times \frac{6^3}{5}\right) = \frac{1}{2} \div \frac{9}{10} = \frac{1}{2_1} \times \frac{10^5}{9} = \frac{5}{9}$

and $\left(\frac{1}{2} \div \frac{3}{4}\right) \div \frac{5}{6} = \left(\frac{1}{2_1} \times \frac{4^2}{3}\right) \div \frac{5}{6} = \frac{2}{3} \div \frac{5}{6} = \frac{2}{3_1} \times \frac{6^2}{5} = \frac{4}{5}$.

$\therefore \quad \frac{1}{2} \div \left(\frac{3}{4} \div \frac{5}{6}\right) \neq \left(\frac{1}{2} \div \frac{3}{4}\right) \div \frac{5}{6}.$

Solved Examples

EXAMPLE 1 **Divide.**

(i) $2\frac{4}{7}$ **by** $1\frac{1}{35}$ **(ii)** $7\frac{1}{7}$ **by** -25 **(iii)** $\frac{-9}{10}$ **by** $\frac{3}{5}$ **(iv)** $\frac{-13}{15}$ **by** $\frac{-26}{75}$

Solution (i) $2\frac{4}{7} \div 1\frac{1}{35} = \frac{18}{7} \div \frac{36}{35} = \frac{18}{7} \times \frac{35}{36} = \frac{18 \times 35}{7 \times 36} = \frac{5}{2} = 2\frac{1}{2}$.

(ii) $7\frac{1}{7} \div (-25) = \frac{50}{7} \div (-25) = \frac{50}{7} \times \left(-\frac{1}{25}\right) = \frac{-50 \times 1}{7 \times 25} = \frac{-2}{7}$.

(iii) $\frac{-9}{10} \div \frac{3}{5} = \frac{-9}{10} \times \frac{5}{3} = \frac{-9 \times 5}{10 \times 3} = \frac{-3}{2}$.

(iv) $\left(\frac{-13}{15}\right) \div \left(\frac{-26}{75}\right) = \frac{-13}{15} \times \frac{-75}{26} = \frac{13 \times 75}{15 \times 26} = \frac{5}{2} = 2\frac{1}{2}$.

EXAMPLE 2 **The product of two rational numbers is $\frac{-15}{22}$. If one of the numbers is $\frac{-9}{44}$, find the other number.**

Solution Let the other rational number be x.

$\therefore \quad x \times \frac{-9}{44} = \frac{-15}{22}$

$\Rightarrow \quad x = \frac{-15}{22} \div \frac{-9}{44} = \frac{-15}{22} \times \frac{-44}{9} = \frac{15 \times 44}{22 \times 9} = \frac{10}{3}$.

Hence, the required rational number is $\frac{10}{3}$.

EXAMPLE 3 **By what number should $\frac{-5}{3}$ be divided to get $\frac{20}{21}$?**

Solution Let the required number be x.

$\therefore \quad \frac{-5}{3} \div x = \frac{20}{21} \Rightarrow \frac{-5}{3} \times \frac{1}{x} = \frac{20}{21}$

$\Rightarrow \quad \frac{1}{x} = \frac{20}{21} \div \frac{-5}{3} = \frac{20}{21} \times \frac{-3}{5} = \frac{-20 \times 3}{21 \times 5} = \frac{-4}{7}$.

$\therefore \quad x = \frac{-7}{4}$.

Hence, the required number is $\frac{-7}{4}$.

EXAMPLE 4 **Fill in the blank.**

$\ldots\ldots \div \frac{2}{3} = -\frac{7}{12}$.

Solution Let $x \div \frac{2}{3} = \frac{-7}{12}$.

$\therefore \quad x \times \frac{3}{2} = \frac{-7}{12}$

$\Rightarrow \quad x = \frac{-7}{12} \div \frac{3}{2} = \frac{-7}{12} \times \frac{2}{3} = \frac{-7 \times 2}{12 \times 3} = \frac{-7}{18}$.

$\therefore \quad -\frac{7}{18} \div \frac{2}{3} = -\frac{7}{12}$.

Here, $-\frac{7}{18}$ fills the blank.

EXERCISE 1C

1. Find.

(i) $\frac{1}{2} \div \frac{3}{4}$ (ii) $\frac{2}{3} \div \frac{-1}{4}$ (iii) $\frac{-4}{5} \div \frac{-5}{4}$ (iv) $\frac{12}{25} \div \frac{-4}{-5}$

(v) $\frac{-6}{-35} \div \frac{15}{14}$ (vi) $\frac{-75}{98} \div \frac{-45}{42}$ (vii) $\frac{-27}{64} \div \frac{4}{-3}$ (viii) $\frac{125}{-32} \div \frac{25}{72}$

(ix) $\frac{3}{-5} \div \frac{-9}{50}$ (x) $\frac{-45}{16} \div 3$ (xi) $16 \div \frac{-64}{25}$ (xii) $-25 \div \frac{5}{-7}$

2. Divide.

(i) $1\frac{1}{7}$ by $5\frac{1}{3}$ (ii) $4\frac{1}{6}$ by -50 (iii) $-36 \div 3\frac{3}{5}$ (iv) $9\frac{4}{5}$ by $16\frac{1}{3}$

3. Verify whether the following statements are true or false.

(i) $\frac{2}{3} \div \frac{-1}{5}$ is a rational number. (ii) $-5 \div \frac{-5}{7}$ is a rational number.

(iii) $\frac{7}{10} \div \frac{3}{2} = \frac{3}{2} \div \frac{7}{10}$.

(iv) $\frac{-32}{11} \div \frac{-16}{-22} \neq \frac{-16}{-22} \div \frac{-32}{11}$.

(v) $\frac{1}{7} \div \left(\frac{-3}{5} \div \frac{1}{3}\right) = \left(\frac{1}{7} \div \frac{-3}{5}\right) \div \frac{1}{3}$.

(vi) $\frac{3}{8} \div \left(\frac{5}{-6} \div \frac{-2}{5}\right) = \left(\frac{3}{8} \div \frac{5}{-6}\right) \div \frac{-2}{5}$.

4. (a) The product of two rational numbers is $\frac{-12}{13}$. If one of them is $\frac{8}{15}$, find the other.
   (b) The product of two rational numbers is -6. If one of the numbers is -8, find the other.
   (c) The product of two rational numbers is 8. If one of the numbers is $\frac{-2}{3}$, find the other.
5. (a) By what rational number should $\frac{-8}{25}$ be divided to get $\frac{3}{10}$?
   (b) By what number should $\frac{12}{13}$ be divided to get $\frac{2}{3}$?
   (c) By what number should -4 be divided to get $\frac{8}{15}$?
6. Divide the sum of $\frac{2}{3}$ and $\frac{-1}{7}$ by their difference.
7. Divide the sum of $\frac{5}{8}$ and $\frac{13}{12}$ by the product of $\frac{2}{3}$ and $\frac{27}{8}$.
8. Fill in the blanks.
   (i) $\frac{8}{27} \div \ldots\ldots = \frac{-4}{9}$. (ii) $\ldots\ldots \div \frac{6}{5} = -\frac{20}{33}$. (iii) $\ldots\ldots \div (-8) = \frac{-25}{12}$. (iv) $(-36) \div \ldots\ldots = \frac{-12}{7}$.
   (v) $\frac{-4}{5} \div \ldots\ldots = -1$. (vi) $\ldots\ldots \div \frac{19}{17} = -1$. (vii) $\frac{3}{8} \div \ldots\ldots = -2$. (viii) $\ldots\ldots \div \left(-\frac{4}{25}\right) = 3$.

ANSWERS

1. (i) $\frac{2}{3}$ (ii) $\frac{-8}{3}$ (iii) $\frac{16}{25}$ (iv) $\frac{3}{5}$ (v) $\frac{4}{25}$ (vi) $\frac{5}{7}$ (vii) $\frac{81}{256}$ (viii) $\frac{-45}{4}$ (ix) $\frac{10}{3}$ (x) $\frac{-15}{16}$ (xi) $\frac{-25}{4}$ (xii) 35
2. (i) $\frac{3}{14}$ (ii) $-\frac{1}{8}$ (iii) -10 (iv) $\frac{3}{5}$
3. (i) True (ii) True (iii) False (iv) True (v) False (vi) False
4. (a) $\frac{-45}{26}$ (b) $\frac{3}{4}$ (c) -12
5. (a) $\frac{-16}{15}$ (b) $\frac{18}{13}$ (c) $\frac{-15}{2}$
6. $\frac{11}{17}$
7. $\frac{41}{54}$
8. (i) $\frac{-2}{3}$ (ii) $\frac{-8}{11}$ (iii) $\frac{50}{3}$ (iv) 21 (v) $\frac{4}{5}$ (vi) $\frac{-19}{17}$ (vii) $\frac{-3}{16}$ (viii) $\frac{-12}{25}$

Word Problems

Solved Examples

EXAMPLE 1 **From a $7\frac{4}{5}$-m-long rope, two pieces of lengths $1\frac{2}{5}$ m and $4\frac{3}{10}$ m are cut off. Find the length of the remaining rope.**

Solution Total length of the two pieces $= 1\frac{2}{5}$ m $+ 4\frac{3}{10}$ m

$$= \left(\frac{7}{5} + \frac{43}{10}\right) \text{m} = \frac{14+43}{10} \text{ m} = \frac{57}{10} \text{ m}.$$

∴ length of the remaining rope $= 7\frac{4}{5}$ m $- \frac{57}{10}$ m

$$= \left(\frac{39}{5} - \frac{57}{10}\right) \text{m} = \frac{78-57}{10} \text{ m}$$

$$= \frac{21}{10} \text{ m} = 2\frac{1}{10} \text{ m}.$$

Hence, the length of the remaining rope is $2\frac{1}{10}$ m.

EXAMPLE 2 **A bag contains four types of fruits weighing $25\frac{1}{4}$ kg in all. If $8\frac{1}{2}$ kg be apples, $7\frac{1}{8}$ kg be oranges, $5\frac{1}{5}$ kg be mangoes, and the rest be grapes, find the weight of the grapes in the bag.**

Solution Total weight of apples, oranges and mangoes

$$= 8\frac{1}{2} \text{ kg} + 7\frac{1}{8} \text{ kg} + 5\frac{1}{5} \text{ kg} = \left(\frac{17}{2} + \frac{57}{8} + \frac{26}{5}\right) \text{kg}$$

$$= \frac{340+285+208}{40} \text{ kg} = \frac{833}{40} \text{ kg} = 20\frac{33}{40} \text{ kg}.$$

So, weight of grapes $= 25\frac{1}{4}$ kg $- 20\frac{33}{40}$ kg

$$= \left(\frac{101}{4} - \frac{833}{40}\right) \text{kg}$$

$$= \frac{1010-833}{40} \text{ kg} = \frac{177}{40} \text{ kg} = 4\frac{17}{40} \text{ kg}.$$

Hence, the weight of the grapes in the bag is $4\frac{17}{40}$ kg.

EXAMPLE 3 **Find the cost of $6\frac{3}{10}$ metres of cloth at the rate of ₹$62\frac{1}{2}$ per metre.**

Solution The cost of $6\frac{3}{10}$ m of cloth at ₹$62\frac{1}{2}$ per metre is

$$6\frac{3}{10} \times ₹62\frac{1}{2} = ₹\left(\frac{63}{10} \times \frac{125}{2}\right) = ₹\left(\frac{63 \times 25}{4}\right) = \frac{₹1575}{4} = ₹393\frac{3}{4}.$$

Hence, the cost of the required length of cloth is ₹$393\frac{3}{4}$.

EXAMPLE 4 **The cost of $5\frac{1}{10}$ L of petrol is ₹$344\frac{1}{4}$. Find the cost of 1 L of petrol.**

Solution The cost of $5\frac{1}{10}$ L $= \frac{51}{10}$ L of petrol is ₹$344\frac{1}{4} = \frac{₹1377}{4}$.

$\therefore$ the cost of 1 L of petrol is $\frac{₹1377}{4} \div \frac{51}{10} = \frac{₹1377}{4} \times \frac{10}{51}$

$= ₹\left(\frac{27 \times 5}{2}\right) = \frac{₹135}{2} = ₹67\frac{1}{2}$.

Hence, the cost of 1 L of petrol is $₹67\frac{1}{2}$.

EXERCISE 1D

1. Ajay cut a $19\frac{3}{5}$-m-long rope into two pieces of lengths $3\frac{1}{4}$ m and $8\frac{1}{2}$m. Find the length of the remaining rope.

2. A metallic container full of wheat weighs $27\frac{1}{5}$ kg. If the empty container weighs $8\frac{1}{4}$ kg, find the weight of wheat in the container.

3. A bag contains $3\frac{1}{5}$ kg of apples, $6\frac{1}{2}$ kg of oranges, $7\frac{3}{10}$ kg of grapes, and some mangoes. If the total weight of the fruits is $25\frac{1}{10}$ kg, find the weight of the mangoes in the bag.

4. One day a cobbler earned $₹140\frac{1}{4}$. He spent $₹16\frac{1}{5}$ on his breakfast and $₹35\frac{1}{2}$ on his lunch, and gave $₹85\frac{1}{4}$ to his wife. Find the remaining amount.

5. Find the cost of $6\frac{1}{5}$ metres of cloth at the rate of $₹30\frac{1}{2}$ per metre.

6. A car covers $21\frac{1}{4}$ km with one litre of petrol. How much distance will it cover with $5\frac{1}{5}$ litres of petrol?

7. The speed of a car is $60\frac{1}{2}$ km/h. Find the distance covered in $4\frac{1}{5}$ hours.

8. Find the area of a square whose side is $4\frac{1}{5}$ m.

9. Find the area of a rectangular farm whose length is $47\frac{2}{5}$ m and breadth is $36\frac{1}{4}$ m.

10. A train covered $901\frac{4}{5}$ km at the speed of $125\frac{1}{4}$ km/h. Find the time taken in covering the distance.

11. A wire of length $42\frac{1}{2}$ m is cut into 34 pieces of equal length. Find the length of each piece.

12. If two thirds of a rational number exceeds its three fifths by $1\frac{3}{4}$, find the rational number.

13. Ravi was asked to multiply a rational number by $\frac{2}{3}$. By mistake he multiplied the rational number by $\frac{2}{9}$. His answer was $-3\frac{1}{5}$ more than the correct answer. Find the rational number.

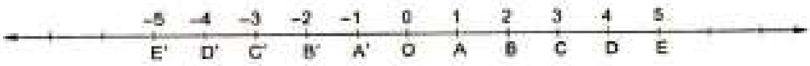
14. From a $62\frac{1}{2}$-m-long rope a piece of length $15\frac{1}{5}$ m is cut off. The rest of the rope is divided into 11 equal pieces. Find the length of each equal piece.

ANSWERS

1. $7\frac{17}{20}$ m	**2.** $18\frac{19}{20}$ kg	**3.** $8\frac{1}{10}$ kg	**4.** ₹$3\frac{3}{10}$
5. ₹$189\frac{1}{10}$	**6.** $110\frac{1}{2}$ km	**7.** $254\frac{1}{10}$ km	**8.** $17\frac{16}{25}$ m^2
9. $1718\frac{1}{4}$ m^2	**10.** $7\frac{1}{5}$ h	**11.** $1\frac{1}{4}$ m	**12.** $\frac{105}{4}$
13. $\frac{36}{5}$	**14.** $4\frac{3}{10}$ m		

Representation of Rational Numbers on the Number Line

You are already familiar with the representation of rational numbers on the number line. Let us review what you have learnt in the previous class.



Draw a straight line and take a point *O* on the number line. Mark points *A, B, C, D, E,* etc., at equal distances on the right of the point *O*.

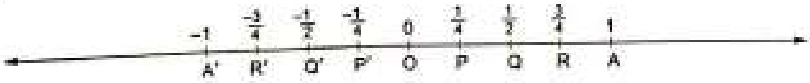
Also, mark points *A', B', C', D', E'*, etc., at the same equal distances on the left of the point *O*.

Let the points *O, A, B, C, D, E,* etc., denote the integers 0, 1, 2, 3, 4, 5, etc., respectively. Similarly, the points *A', B', C', D', E'*, etc., denote the integers −1, −2, −3, −4, −5, etc., respectively.

Thus, all the integers, which are rational numbers too, can be represented on the number line.

EXAMPLE **Represent the rational numbers $\pm\frac{1}{4}$, $\pm\frac{1}{2}$ and $\pm\frac{3}{4}$ on the number line.**

Solution Let the points *O, A* and *A'* represent respectively 0, 1 and −1 on the number line.



First we divide *OA* into four equal parts. Let $OP = PQ = QR = RA = \frac{1}{4}OA$.

So, the points *P, Q* and *R* represent the rational numbers $\frac{1}{4}$, $\frac{2}{4}$ $\left(\text{i.e. } \frac{1}{2}\right)$ and $\frac{3}{4}$ respectively.

Similarly, we divide *OA'* into four equal parts. Let $OP' = P'Q' = Q'R' = R'A' = \frac{1}{4}OA'$.

So, the points P', Q' and R' represent the rational numbers $\frac{-1}{4}$, $\frac{-2}{4}$ $\left(\text{i.e. } -\frac{1}{2}\right)$ and $\frac{-3}{4}$ respectively.

Rational numbers between two rational numbers

There are an infinite number of rational numbers between any two rational numbers.

Let x and y be any two rational numbers such that $x < y$. Then, $\frac{x+y}{2}$ is a rational number between x and y; that is,

$$x < \frac{x+y}{2} < y.$$

EXAMPLE **Find a rational number between $\frac{1}{6}$ and $\frac{1}{7}$.**

Solution A rational number between $\frac{1}{6}$ and $\frac{1}{7}$ is $\frac{\frac{1}{6}+\frac{1}{7}}{2} = \frac{13}{84}$.

So, the rational number $\frac{13}{84}$ lies between $\frac{1}{6}$ $\left(\text{i.e. } \frac{14}{84}\right)$ and $\frac{1}{7}$ $\left(\text{i.e. } \frac{12}{84}\right)$.

EXAMPLE **Find two rational numbers between $\frac{3}{5}$ and $\frac{2}{3}$.**

Solution A rational number between $\frac{3}{5}$ and $\frac{2}{3}$ is $\frac{\frac{3}{5}+\frac{2}{3}}{2} = \frac{19}{30}$.

A rational number between $\frac{19}{30}$ and $\frac{2}{3}$ is $\frac{\frac{19}{30}+\frac{2}{3}}{2} = \frac{39}{60} = \frac{13}{20}$.

Hence, two such rational numbers are $\frac{19}{30}$ and $\frac{13}{20}$.

Note There are innumerable rational numbers between any two rationals.

Methods of inserting a large number (n) of rational numbers between two given rational numbers

Case I When the denominators of the given rational numbers are the same, take the following steps.

Steps
1. Multiply the numerator and the denominator of each rational number by a number greater than the number n of rational numbers to be inserted (i.e. by $n+1$ or $n+2$ or $n+3$ or ...).
2. Take any n number of integers between the numerators obtained in Step 1.
3. Write the n integers obtained in Step 2 as the numerators and the new denominator obtained in Step 1 as the common denominator to obtain the required n number of rational numbers between the given rational numbers.

EXAMPLE **Find three rational numbers between $\frac{1}{5}$ and $\frac{2}{5}$.**

Solution The given rational numbers have the same denominator (5). To find $n = 3$ rational numbers between the given rationals, we can multiply their numerators and denominators by any integer greater than 3 (i.e. by 4 or 5 or 6 or more).

Let us multiply them by $n + 1 = 4$.

Thus, $\frac{1}{5} = \frac{1 \times 4}{5 \times 4} = \frac{4}{20}$ and $\frac{2}{5} = \frac{2 \times 4}{5 \times 4} = \frac{8}{20}$.

The integers lying between the numerators 4 and 8 are 5, 6 and 7.

So, three rational numbers lying between $\frac{4}{20}\left(\text{i.e. } \frac{1}{5}\right)$ and $\frac{8}{20}\left(\text{i.e. } \frac{2}{5}\right)$ are $\frac{5}{20}\left(\text{i.e. } \frac{1}{4}\right), \frac{6}{20}\left(\text{i.e. } \frac{3}{10}\right)$ and $\frac{7}{20}$.

Alternatively, let us multiply the numerators and the denominators by $n + 4 = 7$.

Then, $\frac{1}{5} = \frac{1 \times 7}{5 \times 7} = \frac{7}{35}$ and $\frac{2}{5} = \frac{2 \times 7}{5 \times 7} = \frac{14}{35}$.

The integers lying between 7 and 14 are 8, 9, 10, 11, 12 and 13.

So, some rational numbers lying between $\frac{7}{35}\left(\text{i.e. } \frac{1}{5}\right)$ and $\frac{14}{35}\left(\text{i.e. } \frac{2}{5}\right)$ are $\frac{8}{35}, \frac{9}{35}, \frac{10}{35}\left(\text{i.e. } \frac{2}{7}\right), \frac{11}{35}, \frac{12}{35}$ and $\frac{13}{35}$. Out of these, any three may be taken as the required rational numbers.

Case II **When the given rational numbers have different denominators,** take the following steps.

Steps

1. Find the LCM of the denominators of the given rational numbers.
2. Express each rational number as a rational number with the LCM as the denominator.
3. Find n number of integers between the numerators of the rational numbers obtained in Step 2.
4. Consider the n integers obtained in Step 3 as the numerators and the LCM obtained in Step 1 as the common denominator of the required rational numbers between the two given rational numbers.

EXAMPLE **Find five rational numbers between $-\frac{3}{5}$ and $\frac{3}{7}$.**

Solution The given rational numbers have different denominators (5 and 7).

The LCM of 5 and 7 is 35.

$\therefore \quad -\frac{3}{5} = \frac{-3 \times 7}{5 \times 7} = \frac{-21}{35}$ and $\frac{3}{7} = \frac{3 \times 5}{7 \times 5} = \frac{15}{35}$.

Then, the numerators become -21 and 15.

The integers lying between -21 and 15 are $-20, -19, -18, -17, \ldots, 0, 1, 2, \ldots, 13, 14$.

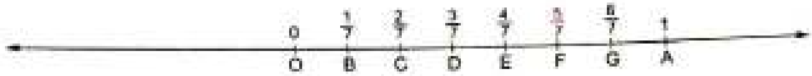
Hence, some rational numbers lying between $\frac{-21}{35}$ $\left(\text{i.e. } \frac{-3}{5}\right)$ and $\frac{15}{35}$ $\left(\text{i.e. } \frac{3}{7}\right)$ are $\frac{-20}{35}, \frac{-19}{35}, \frac{-18}{35}, \frac{-17}{35}, \ldots, 0, \frac{1}{35}, \frac{2}{35}, \ldots, \frac{13}{35}, \frac{14}{35}$. Out of these, any five can be chosen as the required rational numbers.

Solved Examples

EXAMPLE 1 **Represent the following rational numbers on the number line.**

(i) $\frac{5}{7}$ **(ii)** $\frac{-7}{3}$ **(iii)** $3\frac{1}{4}$

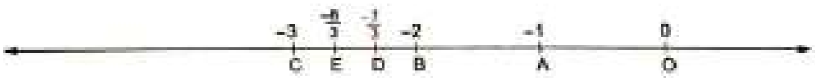
Solution (i) Obviously, $\frac{5}{7}$ lies between 0 and 1. Let the points O and A denote 0 and 1 respectively on the number line.



Divide OA into seven equal parts. Let $OB = BC = CD = DE = EF = FG = GA = \frac{1}{7}OA$.

So, the point F denotes $\frac{5}{7}$ on the above number line.

(ii) $\frac{-7}{3} = -2 + \left(\frac{-1}{3}\right)$. It lies between −2 and −3.

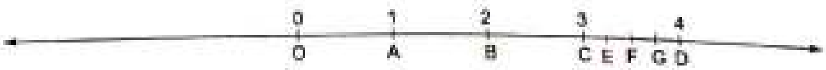


Let the points O, A, B and C denote the integers 0, −1, −2 and −3 respectively on the above number line.

Divide BC into three equal parts. Let $BD = DE = EC = \frac{1}{3}BC$.

So, the point D on the above number line represents $\frac{-7}{3}$.

(iii) $3\frac{1}{4} = 3 + \frac{1}{4}$. It lies between 3 and 4.



Let the points O, A, B, C and D represent the integers 0, 1, 2, 3 and 4 respectively on the above number line.

Divide OD into four equal parts. Let $CE = EF = FG = GD = \frac{1}{4}CD$.

So, the point E on the above number line denotes $3\frac{1}{4}$.

EXAMPLE 2 **Find ten rational numbers between −2 and −3.**

Solution We can write that $-2 = \frac{-30}{15}$ and $-3 = \frac{-45}{15}$.

Hence, some rational numbers lying between −2 and −3 are $\frac{-31}{15}, \frac{-32}{15}, \frac{-33}{15}, \frac{-34}{15}, \ldots, \frac{-43}{15}, \frac{-44}{15}$. Out of these, any ten can be taken as the required rational numbers.

EXAMPLE 3 **Find 15 rational numbers between $\frac{-3}{4}$ and $\frac{5}{6}$.**

Solution The LCM of 4 and 6 is 12.

$\therefore \quad \frac{-3}{4} = \frac{-3 \times 3}{4 \times 3} = \frac{-9}{12}$ and $\frac{5}{6} = \frac{5 \times 2}{6 \times 2} = \frac{10}{12}$.

So, some rational numbers lying between $\frac{-3}{4}$ and $\frac{5}{6}$ are $\frac{-8}{12}, \frac{-7}{12}, \frac{-6}{12}, \ldots, \frac{0}{12}, \frac{1}{12}, \frac{2}{12}, \ldots, \frac{9}{12}$. We can take any 15 rational numbers out of these.

EXERCISE 1E

1. Represent the following numbers on the number line.

(i) $\frac{1}{3}$ (ii) $\frac{3}{5}$ (iii) $\frac{7}{4}$ (iv) $2\frac{3}{5}$

(v) $\frac{-4}{-3}$ (vi) $3\frac{1}{4}$ (vii) $4\frac{1}{2}$ (viii) $5\frac{2}{3}$

2. Represent each of the following numbers on the number line.

(i) $\frac{-2}{3}$ (ii) $\frac{-4}{5}$ (iii) $\frac{-9}{4}$ (iv) $-3\frac{1}{2}$

(v) $\frac{-5}{3}$ (vi) $-2\frac{1}{3}$ (vii) $-1\frac{1}{5}$ (viii) $-4\frac{1}{4}$

3. Find a rational number between $\frac{1}{2}$ and $\frac{1}{3}$.

4. Find two rational numbers between $\frac{3}{4}$ and $\frac{1}{7}$.

5. Find three rational numbers between $\frac{1}{4}$ and $\frac{3}{4}$.

6. Find four rational numbers between $\frac{1}{7}$ and $\frac{3}{7}$.

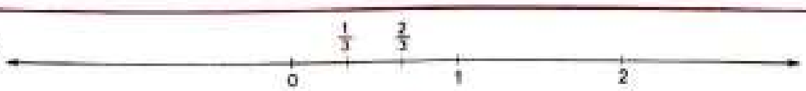
7. Find five rational numbers between −3 and −4.

8. Find ten rational numbers between −4 and −5.

9. Find 20 rational numbers between $\frac{-7}{12}$ and $\frac{7}{9}$.

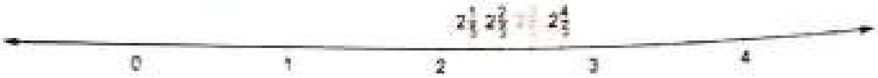
ANSWERS

1. (i)



(ii)

(iii) 

(iv) 

(v) 

(vi) 

(vii) 

(viii) 

2. (i) 

(ii) 

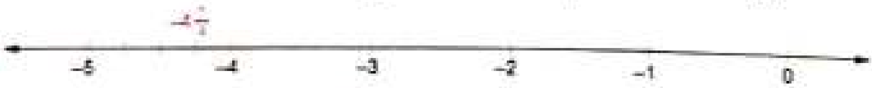
(iii) 

(iv) 

(v) 

(vi) 

(vii) 

(viii) 

3. $\frac{5}{12}$ **4.** $\frac{25}{56}, \frac{33}{112}$ **5.** $\frac{3}{8}, \frac{4}{8}\left(\text{i.e.} \frac{1}{2}\right), \frac{5}{8}$ **6.** $\frac{4}{21}, \frac{5}{21}, \frac{6}{21}, \frac{7}{21}$

7. $\frac{-19}{6}, \frac{-20}{6}, \frac{-21}{6}, \frac{-22}{6}, \frac{-23}{6}$

8. $\frac{-45}{11}, \frac{-46}{11}, \frac{-47}{11}, \frac{-48}{11}, \frac{-49}{11}, \frac{-50}{11}, \frac{-51}{11}, \frac{-52}{11}, \frac{-53}{11}, \frac{-54}{11}$

9. $\frac{-20}{36}, \frac{-19}{36}, \frac{-18}{36}, \frac{-17}{36}, \frac{-16}{36}, \frac{-15}{36}, \frac{-14}{36}, \frac{-13}{36}, \frac{-12}{36}, \frac{-11}{36}, \frac{-10}{36}, \frac{-9}{36}, \frac{-8}{36}, \frac{-7}{36}, \frac{-6}{36}, \frac{-5}{35}, 0, \frac{1}{36}, \frac{2}{36}, \frac{27}{36}$

Note that there are an infinite number of rationals between any two rational numbers. Hence, the student's answers to the questions **3–9** *may differ from our answers.*

❖